HIGH SCHOOL ALGEBRA

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TANNER'S HIGH SCH. ALG.

W. P. I.

PREFACE

In the preparation of this book the author's aim has been:

(1) To make the transition from arithmetic to algebra as easy and natural as possible, and to arouse the pupil's interest by showing him early some of the advantages of algebra over arithmetic.

(2) To present the several topics in the order of their simplicity, giving definitions only where they are needed, and insuring clearness of comprehension by an abundance of concrete illustrations and inductive questions.

(3) To provide a large, well-chosen, and carefully graded set of exercises, the solution of which will help not only to fix in the pupil's mind the principles involved, but also further to unfold those principles.

(4) To omit non-essentials, and yet provide a book that fully meets the entrance requirements in elementary algebra of any college or university in this country.

Among other features of this book to which attention is invited are: (1) the careful statement of definitions and principles; (2) the emphasis laid upon translating formulas and equations into verbal language, and *vice versa;* (3) the inclusion of many formulas from physics which the pupils are asked to solve for the various letters which they contain; and (4) the extensive cross-references, as well as the many "hints" and "suggestions" found among the exercises and problems, all calculated to throw sidelights upon the work.

On the request of several prominent mathematics teachers the author has put an elementary chapter on quadratic equations (Chap. XII) before the chapters on radicals, imaginaries, and the theory of exponents. This arrangement is made possible by the treatment of radicals of the second order given in § 121 (only such

radicals are met with in Chap. XII), and it has important pedagogical as well as practical advantages over the more usual arrangement. Those teachers, however, who prefer the usual order, may omit § 121 altogether, and take Chaps. XIV and XV, except §§ 162, 163, and 170, before taking Chap. XII.

In order to avoid unnecessary repetition, the work on graphs has nearly all been collected into a single chapter. This arrangement has made it possible to give this topic a somewhat more adequate treatment than is usual in a book of this kind, and to do so without giving it more than its rightful amount of space. By this arrangement also those schools which do not take graphs in their first year's work will find their algebra work uninterrupted, while appropriately placed footnotes indicate the connection in which the parts of this chapter may be most advantageously read by those who wish to include graphs.

Scattered through the book are a few articles (marked with a *) which should be taken in connection with the review work when time permits; the omission of these articles, however, does not anywhere break the continuity of the work. For the benefit of the brightest pupils in a class there have been inserted here and there references to the author's Elementary Algebra, where the topics concerned are discussed somewhat more fully; a few copies of the Elementary Algebra placed in the school library can in this way be made to serve a very useful purpose.

It is with great pleasure that the author acknowledges his indebtedness to the many experienced teachers of Algebra in High Schools and Academies in all quarters of our country whose suggestions to him have added so much of value to this book. Special acknowledgments are due to Prof. J. M. McPherron of the Los Angeles High Schools for reviewing the manuscript before it went to press, and to Miss Cora Strong of the State Normal School of Greensboro, N.C., who secured a leave of absence from her school so that she might give her entire time to assisting in the work on this book; to her mainly belongs the credit for the excellent exercises which are found in the following pages.

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HIGH SCHOOL ALGEBRA

CHAPTER I

INTRODUCTION

I. LITERAL NUMBERS

1. Algebra. In the following pages, we shall continue to use the symbols 0, 1, 2, 3, etc. to represent numbers, and the signs $+$, $-$, $\times$, $\div$, and $=$, to denote addition, subtraction, multiplication, etc.; that is, we shall use all of these characters just as we used them in arithmetic. We shall presently see, however, that algebra greatly simplifies the solution of certain kinds of problems (§ 3), that it introduces new kinds of number (§§ 13, 146, 164), and that it makes extensive use of letters to represent numbers.

2. Numbers represented by letters. In arithmetic, numbers are almost always expressed by means of the symbols 0, 1, 2, 3, etc., but letters also are sometimes used. For example, in interest problems p often stands for *principal*, r for *rate*, t for *time*, i for *interest*, and a for *amount*.

In algebra, on the other hand, the use of letters to represent numbers is very common; thus, just as in arithmetic we speak of 4 books, 7 bicycles, 85 pounds, 3 men, etc., so in algebra we use, not only these expressions, but also such expressions as a books, n bicycles, x pounds, y men, etc. Numbers represented by letters are often called **literal numbers.**

In the case of literal numbers the operations of addition, subtraction, etc. may be indicated just as these operations are indicated with arithmetical numbers. Thus, for example,

if n represents one number and k another, then $n+k$ stands for their sum, $n-k$ for their difference, $n \times k$ for their product, $n \div k$ or $\frac{n}{k}$ for their quotient.

EXERCISE I

If a stands for 3, b for 2, and x for 12, find the value of each of the following expressions:

1. $a+b$.

2. $x-a$.

3. $x \div a$.

4. $5b-\frac{x}{a}$.*

5. $\frac{2a+x}{b}$.

6. $\frac{a+bx}{3a}$.

7. $\frac{ab+2x-10}{4a+4b}$.

8. $\frac{3abx-ab}{ab+bx}$.

9. $\frac{x}{a}+\frac{4a}{2b}-\frac{x}{2}$.

10. $\frac{b}{a}+\frac{x-3a}{b}$.

11. If s represents 16, what number is represented by $2s$? by $\frac{1}{4}s$, or (as usually written) $\frac{s}{4}$? by $\frac{3s}{8}$?

12. If a suit of clothes costs 8 times as much as a hat, and if h stands for the cost of the hat, how may the cost of the suit be represented?

13. Does $h+8h$ (*i.e.*, $9h$) represent the combined cost of the hat and suit in Ex. 12? Explain your answer.

14. The side of a square is 5 feet long. How long is the bounding line of this square? How long is the bounding line if the side is x feet long?

15. A boy's present age is 15 years; *indicate*, without performing the subtraction, his age 4 years ago. What was his age n years ago? What will it be y years hence?

16. At 5 cents each how many erasers can be bought for 15 cents? for x cents? for n dollars?

17. What number multiplied by 8 gives the product 40? If $8x=40$, what is the value of x? If $5y+2y=21$, what is the value of y?

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* $5b$ means 5 times b; so too ab means a times b; and $3ax$ means the product of 3, a, and x.

3. One advantage of literal numbers. The following examples show how the solution of problems may often be simplified by using letters to represent numbers.

Prob. 1. A gentleman paid \$45 for a suit of clothes and a hat. If the clothes cost 8 times as much as the hat, what was the cost of each?

ARITHMETICAL SOLUTION

The hat cost "a certain sum," and since the clothes cost 8 times as much as the hat, therefore the cost of the clothes was 8 times "that sum," and the cost of the two together was 9 times "that sum." Hence 9 times "that sum" is \$45, and therefore "that sum" is \$5, and 8 times "that sum" is \$40; *i.e.*, the hat cost \$5, and the clothes \$40.

This solution may be put into the following more systematic form, still retaining its arithmetical character.

	A certain sum = the cost of the hat;
then $\because$ *	8 times that sum = the cost of the clothes,
$\therefore$	9 times that sum = the cost of both,
i.e.,	9 times that sum = \$45.
$\therefore$	that sum = \$5, the cost of the hat,
and	8 times that sum = \$40, the cost of the clothes.

ALGEBRAIC SOLUTION

The solution just given becomes much simpler if we let a single letter, say x, stand for the number of dollars in "a certain sum" and "that sum" as used above, thus:

Let	$x =$ the number of dollars the hat cost.
Then	$8x =$ the number of dollars the clothes cost,
and	$x + 8x =$ the number of dollars both cost;
i.e.,	$9x = 45.$
$\therefore$	$x = 5$, and $8x = 40$;

i.e., the hat cost \$5, and the clothes cost \$40.

N.B. The letter x, above, stands for a *number*, not for the cost of the hat.

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* The symbols $\because$ and $\therefore$ stand for the words "since" and "therefore," respectively.

Prob. 2. If a locomotive weighs 3 times as much as a car, and the difference between their weights is 50 tons, what is the weight of the locomotive?

SOLUTION

Let $w =$ the number of tons in the weight of the car.
Then $3w =$ the number of tons in the weight of the locomotive, and, since the difference between their weights is 50 tons,

$\therefore$ $$3w - w = 50,$$
i.e., $$2w = 50,$$
whence $$w = 25,$$
and $$3w = 75;$$
i.e., the locomotive weighs 75 tons.

Prob. 3. Of three numbers the second is 5 times the first, and the third 2 times the first; if the sum of these numbers equals the third number increased by 42, what are the numbers?

SOLUTION

Let $n =$ the first of the three numbers.
Then $5n =$ the second number,
and $2n =$ the third number;
now since the sum of the three equals the third number increased by 42,

$\therefore$ $n + 5n + 2n = 2n + 42,$
i.e., $8n = 2n + 42,$
hence $6n = 42.$ [Subtracting $2n$ from each of the equal sums above.]
$\therefore$ $n = 7,\quad 5n = 35,$ and $2n = 14;$
i.e., the numbers are 7, 35, and 14, respectively.

REMARK. Observe that the *steps* in each of the foregoing solutions are:

1. To let some letter, say x, stand for one of the unknown numbers (preferably the smallest).
2. To express the other unknown numbers in terms of x.
3. To translate into *algebraic language* those relations between the unknown numbers which the problem states in words; this

translation gives an equation, and from it the required numbers are easily found.

Observe also that while the above problems *can* be solved by arithmetic, the algebraic solution is much simpler.

EXERCISE II

Solve the following problems:

4. In a room containing 45 pupils there are twice as many boys as girls. How many boys are there in the room?

5. If a horse costs 7 times as much as a saddle, and if the difference in the cost of the two is \$90, find the cost of each.

6. A house is worth 5 times as much as the lot on which it stands, and the two together are valued at \$4200. Find the value of each.

7. If the house and lot of Ex. 6. *differ* in value by \$4200, how much is each worth?

8. The double of a certain number taken from 10 times the same number leaves 72. What is the number?

9. If n represents a certain number, how may we represent:

(1) the number, plus 4 times itself, plus 5 times itself?

(2) the sum of the number, its double, and its half?

What does $5n + 7n - 3n$ represent?

10. A number, plus twice itself, plus 4 times itself, is equal to 56. What is the number?

11. Divide 98 into three parts such that the second is twice the first and the third is twice the second.

12. Divide 160 into three parts such that two of them are equal, while the third is twice either of the others.

13. In a yachting party consisting of 36 persons, the number of children is 3 times the number of men, and the number of women is one half that of the men and children combined. How many women are there in the party?

14. If I have s nickels, how many cents have I? How many cents in s dimes? in s quarters? in the sum of s nickels, s dimes, and s quarters?

15. A boy found that he had the same number of 5, 10, and 25 cent pieces, and that the total amount of his money was $3.20. How many coins of each kind had he?

16. Alice buys Christmas gifts at 25 cents, 15 cents, and 10 cents — the same number at each price. If she spends $2 in all, how many gifts does she buy?

17. If x stands for a certain number, what would stand for

(1) the double of the number, increased by 7?

(2) the difference between 3 times the number and 8?

18. How would you represent two numbers whose difference is 4? two numbers whose sum is 13?

19. Find the number whose double, with 4 added, equals 46.

20. Find two numbers, differing by 7, whose sum is 35. Also find two numbers whose sum is 65 and whose difference is 15.

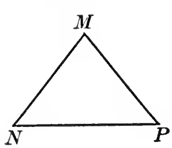
21. William has 8 cents more than his sister Harriet, and together the two have 80 cents. How much money has each? If Harriet's money is made up of an equal number of nickels and one-cent pieces, how many nickels has she?

22. In a family of seven children each child is 2 years older than the next younger. If the sum of their ages is 84 years, how old is the youngest child?

23. A father's age is now 3 times that of his son; 5 years hence, the sum of their ages will be 62 years. Find the present age of each. (Cf. Ex. 15, p. 2.)

24. Four years ago Isabel was twice as old as Mabel; the sum of their present ages is 32 years. How old is each?

25. In a business enterprise, the combined capital of A, B, and C is $21,000. A's capital is twice B's, and B's is twice C's. What is the capital of each?



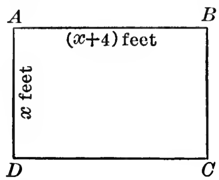
26. In the triangle MNP, NP is 2 inches longer than MN, while PM and MN are of equal length. If the sum of the three sides is 86 inches, find the length of each.

27. An east-bound and a west-bound train leave Chicago at the same hour, the first running twice as fast

as the second; after one hour they are 90 miles apart. Find the speed of each.

28. In a fishing party consisting of four boys, two of the boys caught each the same number of fish, another caught 2 more than this number, and the fourth, 1 less. If the total number of fish caught was 29, how many did each catch?

29. An estate valued at $24,780 is to be divided among a family consisting of a mother, two sons, and three daughters. If the daughters are to receive equal shares, each son twice as much as a daughter, and the mother twice as much as all the children together, what will be the share of each?



30. *ABCD* represents the floor of a room. Find the dimensions of the floor if its bounding line is 48 feet long.

31. A gallon of cream is poured into two pitchers, one of which holds 7 times as much as the other. How many gills does each pitcher hold?

32. If $\frac{1}{3}$ of a number is added to the number, the sum is 120. What is the number?

SUGGESTION. Let $3x =$ the number.

33. If $\frac{1}{3}$ of a number is added to twice the number, the sum is 35. What is the number?

34. Of two numbers, twice the first is 7 times the second, and their difference is 75. Find the numbers.

SUGGESTION. Let $7x =$ the first number, then $2x =$ the second.

35. An estate of $19,600 was so divided between two heirs that 5 times what one received was equal to 9 times what the other received. What was the share of each?

36. A tree whose height was 150 feet was broken off by the wind, and it is found that 3 times the length of the part left standing is the same as 7 times that of the part broken off. How long is each part?

37. If two boys together solved 65 problems, and if 8 times the number solved by the first boy equals 5 times the number solved by the second boy, how many did each boy solve?

II. ELEMENTARY OPERATIONS

4. Addition. In algebra, as in arithmetic, such an expression as $7 + 3$ is read "7 plus 3," and means that 3 is to be added to 7.

To perform this addition we begin at 7 and count 3 forward, obtaining the result 10, which is called the sum of these two numbers.*

So also if a and b stand for any two numbers whatever, the expression $a + b$ is read "a plus b," and means that b is to be added to a.

The result obtained by adding two or more numbers is called their **sum**, and the numbers that are to be added are called the **summands**.

5. Subtraction. To what number must 3 be added to obtain the sum 8? If 8 was obtained by adding 3 to some number (*i.e.*, by counting 3 *forward*), how may we, starting with 8, find the number at which the counting began?

Here, as in arithmetic, the operation of finding this number is indicated by the expression $8 - 3$, which is read "8 minus 3." We may say that $8 - 3 = 5$ *because* $5 + 3 = 8$.

The process of finding one of two numbers when their sum and the other number are given, is called **subtraction.** It consists, as we have just seen, in counting backward, *i.e.*, in *undoing* the work of addition, which consists in counting forward.

If a and b stand for any two numbers whatever, the expression $a - b$ is read "a minus b," and means that b is to be subtracted from a.

The result obtained by subtracting one number from another is called their **difference** (also the **remainder**). The number which is to be subtracted is called the **subtrahend,** and

* If fractions are to be added, we first reduce them to a common denominator and then add their numerators; it is still a counting process.

the one from which the subtraction is to be made is called the **minuend.**

6. Inverse operations. Of two operations which neutralize each other when performed in succession, each is called the **inverse** of the other. Thus the operations of addition and subtraction are each the inverse of the other (cf. Exs. 8–10, below).

EXERCISE III

Read each of the following expressions, then name its parts:

1. $8+12=20$. **2.** $9-7=2$. **3.** $12y-9y=3y$.

4. Since $4+9=13$, therefore $13-9=?$ $13-4=?$

5. In subtraction, what name is used to denote the given sum? the given summand? the required summand? Illustrate, using Ex. 2 above.

6. Add 4 to 7 by counting. Where do you begin to count? In what direction do you count?

7. By counting, subtract 4 from 11. Do you count in the same direction as in Ex. 6?

8. How may you combine the subtrahend and remainder to get the minuend? Why?

9. How would you test the correctness of an answer in subtraction? Illustrate. Could you use subtraction to test the correctness of a sum?

10. When is one operation said to be the inverse of another? Using the numbers 8 and 6, illustrate the fact that subtraction is the inverse of addition.

11. If m and n stand for any two given integers whatever, can you, by counting, find the value of $m+n$? of $m-n$?

12. What is the value of $5-3$? of $5-4$? of $5-5$? of $5-6$? of $5-8$? In order that subtraction be possible, how must the subtrahend compare in size with the minuend?*

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* With our present (arithmetical) meaning of number such a subtraction as $5-8$ is, of course, impossible; in Chapter II, however, we shall so extend the meaning of number as to make the subtraction $a-b$ possible even when b is greater than a.

7. Multiplication. (i) In arithmetic, multiplication is usually defined as the process of taking (additively) one of two numbers, called the multiplicand, as many times as there are units in the other, called the multiplier. In this sense, 6×4 (read "6 multiplied by 4") means $6 + 6 + 6 + 6$; *i.e.*, this multiplication may be regarded as an abbreviated addition.

Strictly speaking, however, the above definition of multiplication applies only when the multiplier is an arithmetical integer: under this definition, for instance, we could not find such a product as $8 \times 5\frac{2}{3}$, because we could not take the multiplicand *two thirds of a time* any more than we could fire a gun two thirds of a time.

(ii) A broader definition of multiplication, and one better suited to our present purpose, may be stated thus:

Multiplication is the process of performing upon one of two given numbers (the **multiplicand**) the same operation as that which is performed upon unity to get the other given number (the **multiplier**); the result thus obtained is called the **product** of these numbers. The multiplicand and multiplier are called **factors** of the product.

To illustrate, consider again the question of multiplying 8 by $5\frac{2}{3}$. The multiplier, $5\frac{2}{3}$, is obtained from unity by taking the unit 5 times, and $\frac{1}{3}$ of the unit twice, as summands, *i.e.*,

$$5\tfrac{2}{3} = 1 + 1 + 1 + 1 + 1 + \tfrac{1}{3} + \tfrac{1}{3};$$

and, therefore, by this new definition of multiplication,

$$8 \times 5\tfrac{2}{3} = 8 + 8 + 8 + 8 + 8 + \tfrac{8}{3} + \tfrac{8}{3} = 40 + \tfrac{16}{3} = 45\tfrac{1}{3}.$$

(iii) Just as 6×5 means that 6 is to be multiplied by 5, so $b \times 3$ means that b is to be multiplied by 3. Similarly, $k \times n \times y$ means that k is to be multiplied by n, and that their product is then to be multiplied by y.

Instead of the oblique cross ($\times$), a center point ($\cdot$) placed between two numbers (a little above the line to distinguish it from the decimal point) is frequently used as a sign of

multiplication. And even the center point is usually omitted if doing so causes no confusion. Thus, $3 \times n = 3 \cdot n = 3n$; so, too, $p \times r \times t = p \cdot r \cdot t = prt$, and $3 \times 7 = 3 \cdot 7$. But the sign (cross or center point) must not be omitted between two arithmetical numbers. (Why not?)

EXERCISE IV

Read each of the following expressions, then name its parts:

1. $8 \times 3 = 24$. **2.** $\frac{2}{3} \cdot 15 = 10$. **3.** $5a \cdot 4 = 20a$.

4. What is the value of $5 \cdot 3$? How is this product obtained under the old definition of multiplication [§ 7 (i)]?

5. Using the new definition [§ 7 (ii)], show that $5 \cdot 3$ means $5 + 5 + 5$. Similarly, explain the meaning of $9 \cdot 4$; of $4 \cdot 9$.

6. Show that $2\frac{3}{4} \cdot 8$ has the same meaning under the old definition of multiplication as under the new.

7. To get $\frac{3}{5}$ from 1, we divide 1 into how many equal parts? How many of these parts do we take? What, then, should be done to 10 in multiplying it by $\frac{3}{5}$?

As in Ex. 7, find the following products:

8. $16 \cdot \frac{3}{4}$. **9.** $12 \cdot 2\frac{2}{3}$ **10.** $7 \cdot 5\frac{3}{5}$.

If $a = 2$, $b = 5$, and $x = \frac{2}{3}$, find the value of each of the following expressions:

11. $3abx$. **13.** $2bx - ax + 3b$. **15.** $3aax + 4bx$.

12. $5b + 6x - ab$. **14.** $7abx + 3a - 2b$. **16.** $aab - 10x$.

8. Division. Division is the inverse (*i.e.*, the "undoing") of multiplication. Thus, since $4 \times 9 = 36$, therefore $36 \div 9 = 4$, and $36 \div 4 = 9$.

The expression $36 \div 9 = 4$ is read "36 divided by 9 equals 4." Here 36 is called the *dividend*, 9 the *divisor*, and 4 the *quotient*.

In multiplication we have given two numbers, and are asked to find their product; in division we have given the product (now called the *dividend*) and one of the factors

(now called the *divisor*), and are asked to find the other factor (now called the *quotient*).

Hence we may say: **division** is the process of finding from two given numbers, called **dividend** and **divisor**, respectively, a third number (called the **quotient**) such that the divisor multiplied by the quotient equals the dividend.

E.g., $36 \div 9 = 4$, *because* $4 \times 9 = 36$.

If s and t represent any two numbers whatever, then each of the expressions, $s \div t$, $\frac{s}{t}$, s/t, and $s : t$ indicates that s is to be divided by t.

If the divisor is not exactly contained in the dividend, then, as in arithmetic, the indicated division is called a **fraction**.

E.g., $\frac{2}{3}$, $\frac{16}{5}$, $\frac{m}{n}$, and $\frac{a + x}{y}$ are called fractions.

It is to be remarked, however, that literal numbers may be fractional in *form* but integral in *value*, and *vice versa*. Thus, $\frac{a}{b}$, though fractional in form, has the integral value 3 if $a = 12$ and $b = 4$.

9. Powers, exponents, etc. (i) In algebra, as in arithmetic, such a product as $5 \cdot 5 \cdot 5$ is usually written in the abbreviated form 5^3, the small 3 showing the number of times that 5 is used as a factor.

Similarly, $2^3 = 2 \cdot 2 \cdot 2$, $a^3 = a \cdot a \cdot a$, $2^3 \cdot 5^2 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5$, $n^2p^4 = n^2 \cdot p^4 = n \cdot n \cdot p \cdot p \cdot p \cdot p$, etc.

The expression k^4 is usually called the *fourth power* of k. In this expression, 4 is called the *exponent* and k the *base* of the power.

(ii) Hence the following definitions: A **power** of a number is the product arising from using the given number one or more times as a factor.

An **exponent** is a number placed (in small symbols) at the

right and slightly above a given number, to show how many times the latter is to be used as a factor.

Thus, if x represents any number whatever, and n any arithmetical integer,* then the expression x^n is called the *nth* power of x, and means the product arising from using x as a factor n times; n is the exponent of the power.

NOTE. Observe that under the above definitions a^1 has the same meaning as a; the exponent 1, therefore, need not be written.

The second and third powers of numbers are, for geometrical reasons, often called by the special names of **square** and **cube** respectively. Thus a^2 is called "the second power of a," "the square of a," and also "a squared."

EXERCISE V

Read each of the following expressions, name its parts, and test the correctness of the results:

1. $18 \div 6 = 3.$

2. $28 \div 14 = 2.$

3. $\frac{630}{9} = 70.$

4. $\dfrac{20\,a}{5} = 4\,a.$

5. $9^2 = 81.$

6. $2^5 \cdot 3^2 = 288.$

7. $\dfrac{8^3}{2^4} = 32.$

8. $\dfrac{12\,a^4}{3^2} = \dfrac{4\,a^4}{3}.$

Read the following expressions and tell what operations are indicated in each case; then find the numerical value of each expression when $a = 5$, $b = 2$, $n = 1$, and $x = 4$.

9. $a^5.$

10. $a^3 + b^3.$

11. $7\,n^4x^3.$

12. $\dfrac{3\,b^5}{16}.$

13. $8^2ab^3 - 10\,x^2.$

14. $n^3 + 5\,ax^2.$

15. $\dfrac{a^3 - 10\,b^3}{3\,a}.$

16. $\dfrac{75 - ab^2n^3}{x^2 - a}.$

17. Write $7 \cdot 7 \cdot 7 \cdot 7$ by means of the exponent notation. Also $a \cdot a \cdot a$; $5 \cdot 5 \cdot x \cdot x \cdot x$; and $9 \cdot 9 \cdot 9 \cdot 9 \cdot a \cdot a \cdot y \cdot y \cdot y$.

18. How may we use multiplication to test the correctness of an example in division? Why?

19. The sum of any two integers is integral. Is this true of their difference? of their product? of their quotient? Illustrate your answers.

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* We shall later (Chapter XVI) enlarge the scope of such a symbol as x^n by giving it a meaning even when n does not represent an arithmetical integer.

20. When the dividend is not exactly divisible by the divisor, what name is given to the indicated quotient?

21. How are fractions defined in arithmetic? Is $\frac{5}{2\frac{3}{4}}$ a fraction under the arithmetical definition? If not, why not?

10. The order in which arithmetical operations are to be performed. What is the value of $2 + 6 \cdot 5 - 8 \div 2$? Is it 28, 16, or 12? In order that such an expression shall have the same meaning for all of us, mathematicians have agreed that, when there is no express statement to the contrary:

(1) A succession of multiplications and divisions shall mean that these operations are to be performed in the order in which they occur from left to right.

(2) A succession of additions and subtractions shall mean that they are to be performed in the order in which they occur.

E.g., $9 \cdot 8 \div 6 \cdot 2 = 72 \div 6 \cdot 2 = 12 \cdot 2 = 24,$

but $9 \cdot 8 \div 6 \cdot 2$ is *not* equal to $72 \div 12$, *i.e.*, to 6.

So, too, $7 + 9 - 6 + 3 = 16 - 6 + 3 = 10 + 3 = 13,$

but $7 + 9 - 6 + 3$ is *not* equal to $16 - 9$, *i.e.*, to 7.

(3) A succession of the operations of addition, subtraction, multiplication, and division shall mean that *all* the operations of multiplication and division are to be performed before *any* of those of addition and subtraction, and in accord with (1) above. The additions and subtractions are then to be performed in accord with (2) above.

E.g., $2 + 6 \cdot 5 - 8 \div 2 = 2 + 30 - 4 = 28.$

Note. While such an expression as $3 \cdot a \div 2 \cdot x \cdot y$ means $[(3\,a) \div 2] \cdot x \cdot y$, the expression $3\,a \div 2\,xy$ is usually understood to mean $(3\,a) \div (2\,xy)$; *i.e.*, $3\,a$ and $2\,xy$ are here understood to represent *products* rather than unperformed multiplications.

11. Signs of aggregation. (i) Any desired departure from the order of operations given in § 10 may be indicated by employing one or more of the so-called **signs of aggregation**;

among these are the **parenthesis** ( ), the **brace** { }, the **bracket** [ ], and the **vinculum** $\overline{\qquad}$.

(ii) An expression within a parenthesis, brace, or bracket, or under a vinculum, is to be regarded as a *whole*, and is to be treated as though it were represented by a single symbol.

E.g., $(2+6)\cdot 5 \div 3 - (7+8\div 2) = 8\cdot 5\div 3 - 11$, *i.e.*, $2\frac{1}{3}$. So, too, $(4+6)\div 2 = 5$, while without the parenthesis its value would be 7.

It may sometimes be useful even to employ one sign of aggregation within another.

E.g., $72 \div \{47 - 7(15-10)\} = 72 \div \{47 - 35\} = 72 \div 12 = 6$.

EXERCISE VI

Find the value of each of the following expressions:

1. $20+5-3$. **4.** $12\div 2\times 4$. **7.** $16\div 2+6$.

2. $20-5+3$. **5.** $12\div(2\times 4)$. **8.** $16\div(2+6)$.

3. $20-(5+3)$. **6.** $9\cdot(6-2)$. **9.** $11\cdot 4 - 6\cdot 3\div 2$.

10. $28-(6+13)-(10-2)$. **13.** $42\div 7\times 5-5+6\times 2$.

11. $32-9+6\div 2+1$. **14.** $12+9\cdot 3-30\div 2+8$.

12. $32-(9+6)\div(2+1)$. **15.** $(12+9)\cdot 3-(30\div 2+8)$.

16. $\{25-(10+13)\}\div 2+31-\overline{5+4}$.

17. $16\cdot 9-4(36\div 3\cdot 2)+54-(17-\overline{12-5})$.

Read each of the following expressions, and tell in what order the indicated operations are to be performed:

18. $ac+b$. **21.** $(c-b)^2$. **24.** $\dfrac{6a\div 2c-2d}{cd^3}$.

19. $a(c+b)$. **22.** $c^2\div b^2-2d$.

20. $c-b^2$. **23.** $c^2\div(b^2-2d)$. **25.** $\dfrac{(b^4-12d)\div\overline{a-b}}{cd^2+[2(c\div a)]^2}$.

26. If $a=8$, $b=3$, $c=12$, and $2d=1$, find the numerical value of each of the expressions in Exs. 18–25.

CHAPTER II

POSITIVE AND NEGATIVE NUMBERS

12. Introductory.* Suppose the present reading of a thermometer is 5° above zero, and the temperature is falling; what will be the reading when it has fallen 1°? 2°? 3°? 4°? 5°? 6°? 7°? 8°?

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* NOTE TO THE TEACHER. It will stimulate interest in the work, as well as enlarge the pupil's view, if the teacher will amplify and present to the class the following considerations (cf. also *El. Alg.* pp. 18, 19):

1. Man's earliest idea of number came from *counting*, and in this way there arose the number system consisting of the integers 1, 2, 3, 4, ... Presently he advanced a step and counted *backward* as well as forward, and by *groups* of things as well as by single things; this led the way to the *operations* of addition, subtraction, multiplication, and division.

2. Having devised this number system and these operations to meet the needs of his daily life (just as later he invented the clock, the steam engine, the telephone, etc.), he soon found the system inadequate: $7 \div 3$, for example, represents no number in the above system. His increasing desire for exactness, however, as civilization advanced, demanded that division should be always possible; for this and other reasons he invented fractions, and included them in his number system.

3. Many of the things with which we are concerned bear a relation of opposition to each other (assets and liabilities, thermometer readings above zero and below zero, etc.), and the change from one of these opposites to the other may be regarded as a subtraction (cf. § 12); for this and other reasons it was found advantageous again to extend the number system, and thus to make subtraction always possible.

4. The use of literal notation, too, quite apart from the foregoing considerations, demands a number system in which addition, subtraction, etc., are always possible. If subtraction, for example, is not always possible, such an expression as $a - b$ may or may not represent a number; hence, should $a - b$ occur in the solution of a problem *before* the relative values of a and b were known, our work would come to a standstill. It is wiser, therefore, to include negatives in our number system, and let the solution proceed, giving the necessary *interpretation* to our results later.

Which one of the arithmetical operations (addition, subtraction, etc.) did you use in answering these questions?

Again, if a business man's financial losses are large enough, they will decrease his capital, not only *to* zero but *through* zero, and bring him into debt.

So, too, if a traveler now in north latitude goes south far enough, he will pass *through* zero latitude, and into south latitude.

Observe that, in each of the statements just made, the change from a given condition to its opposite is essentially *a process of subtraction;* it is a subtraction, moreover, in which we can subtract not only *to*, but *through* zero.

In our present number system, we can subtract only *to* (not *through*) zero; in order, therefore, to express *in the simplest possible way* the numerical relations between such opposite things as those given above (gains and losses, latitude north and south, etc.), we must *extend* our number system so as to make subtraction *through* zero possible.

13. The number system extended. The arithmetical integers arranged in a series increasing by one from left to right, and therefore decreasing by one from right to left, are

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \cdots.$$

Addition is performed by counting toward the right (cf. § 4), and subtraction by counting toward the left, in this series. Moreover, addition is always possible because this series extends *without end* toward the right, and subtraction is arithmetically possible only when the subtrahend is not greater than the minuend because this series is *limited* at the left.

Hence, to make subtraction with arithmetical integers always possible, it is only necessary to continue the above series indefinitely toward the left.

Let the result of subtracting 1 from 1 be designated by 0; of subtracting 1 from 0, by -1; of subtracting 1 from -1,

by -2; of subtracting 1 from -2, by -3, etc.; with these new numbers included, the series becomes

$$\cdots, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, \cdots,$$

which extends without end toward the left as well as toward the right.

Since in this enlarged series each number is less by *one* than the next number at its right (and hence greater by one than the next number at its left), therefore addition and subtraction with arithmetical integers may, as before, be performed by counting toward the right and left respectively.

E.g., to subtract 8 from 5, *i.e.*, to find the number which is 8 less than 5, we begin at 5 and count 8 toward the left, arriving at -3; hence, $5 - 8 = -3$.

Similarly, $4 - 6 = -2$, $4 - 9 = -5$, $11 - 16 = -5$, etc.; hence, besides indicating a particular place in the enlarged number series, -5 also indicates *that the subtrahend is* 5 *greater than the minuend.*

Again, to add 7 to -4, *i.e.*, to find the number which is 7 greater than -4, we begin at -4 and count 7 toward the right, arriving at 3; hence $-4 + 7 = 3$.

NOTE. Such an expression as $4 - 9 = -5$ is, of course, not to be understood to mean that 9 actual units of any kind can be subtracted from 4 such units; 4 of the 9 units may be *immediately* subtracted, leaving the other 5 units to be subtracted later if there is anything from which to subtract; in this sense the number -5 may be said to indicate a *postponed* subtraction, and thus to have a *subtractive quality*; hence the appropriateness of attaching the minus sign to such numbers.

14. Positive and negative numbers. Numbers which are less than zero * are called **negative numbers**; while numbers greater than zero are, for distinction, called **positive numbers.**

In writing positive and negative numbers, the latter are *always* preceded by the minus sign, while positive numbers may be written either with or without the plus sign. Thus,

* "Less than zero" in the sense suggested in § 13.

-5, -2.3, -48, and $-\frac{4}{7}$ are negative numbers, while 3, $+84$, and 1.7 are positive numbers.

Such a number as -2 is read either as *negative two* or as *minus two*, while $+6$ is read as *positive six* or *plus six*.

15. Algebraic numbers, etc. Positive and negative numbers together (including, of course, fractions as well as integers) are often called **algebraic numbers**, while positive numbers alone are called **arithmetical numbers**.

By the **absolute value** of a number is meant merely its *size* without regard to its *quality*; thus -2 and $+2$ have the same absolute value; so also have -17.25 and $+17.25$.

Two numbers which have the same absolute value but which are of opposite quality, are called **opposite numbers**; such, for example, are 5 and -5.

NOTE. The signs $+$ and $-$ as used above are called **signs of quality**. We shall continue, however, to use them as signs of operation also; hence it is sometimes necessary, in order to avoid possible confusion arising from this double use, to inclose a number with its quality sign in a parenthesis (cf. Ex. 20, p. 21).

EXERCISE VII

By *counting* along the algebraic number series given in § 13:

1. Add 3 to 8.

2. Add 7 to -3.

3. Add 4 to -4.

4. Add 5 to -12.

5. Subtract 3 from 8.

6. Subtract 7 from 7.

7. Subtract 7 from 4.

8. Subtract 5 from -3.

9. If temperature above zero is indicated by positive numbers, how may temperature below zero be indicated?

10. Interpret the following temperature record taken from a U. S. Weather Bureau report: Albany, $+8°$; Bismarck (S.D.), $-11°$; Buffalo, $-2°$; Denver, $-5°$; and Galveston, $+34°$.

11. In the above record how much warmer is it at Albany than at Bismarck? at Buffalo than at Denver? at Buffalo than at Galveston? Illustrate your answers by diagrams.

12. The total value of a man's property (his assets) is \$15,860, and his total indebtedness (liabilities) is \$1420. What is the net value of his estate? What is the net value of an estate of which the assets are a dollars and the liabilities b dollars?

13. If b exceeds a in Ex. 12, is $a-b$ positive or negative? How should $a-b$ be interpreted in this case? What is meant by saying that the net value of an estate is $-$ \$750?

14. If assets are indicated by positive numbers, interpret the financial conditions indicated by the following numbers: \$783; $-$ \$2568; $-$ \$374.20; and \$(856–1232).

15. Regarding longitude west of Greenwich as positive, indicate by a number and sign that a place is: (1) in 24° east longitude; (2) on the meridian of Greenwich; (3) in 10° west longitude; (4) in 15° east longitude.

16. At 6 A.M. a thermometer records 15° below zero; by noon it has risen 32°. Indicate by a number and sign the reading at 6 A.M. and at noon. Illustrate by a diagram.

17. If on the line $X'OX$ distances to the left of O are called negative, locate the point on this line whose distance from O is: $+.3$ in.; $-.5$ in.; $+.1$ in.

18. If positive numbers are used to denote assets, gains in business, increase of any kind, temperature above zero, easterly motion, south latitude, west longitude, distance down stream, etc., what are the corresponding meanings to be attached to negative numbers?

19. An ocean steamer is in 12° east longitude. If east longitude is indicated by positive numbers, and if the vessel moves westward through 7° of longitude per day, indicate by a number and sign the longitude of the vessel 4 days hence; $1\frac{1}{2}$ days hence; 2 days ago. Illustrate by a drawing.

20. What number must be added to -12 to make the sum 4? Which, then, is the greater, -12 or 4? How much greater? Which of these numbers has the greater absolute value?

16. Addition of algebraic numbers. As we have already seen (§§ 4, 13), to add a positive number to any given number, we begin at the given number and count *forward.*

Moreover, since a negative number represents an unperformed subtraction (§ 13, Note), therefore *adding* a negative number means *performing* this subtraction. Hence, to add a *negative* number to any given number, we begin at the given number and count *backward*.

E.g., to add -6 to 54 we *begin* at 54 and count 6 *backward; i.e.*, $54 + (-6) = 54 - 6 = 48$ (cf. Exs. 19, 20, below).

Such an expression as $11 - 4 - 8 + 3$, which, by what is said above, equals $11 + (-4) + (-8) + 3$, is usually spoken of as an **algebraic sum**.

EXERCISE VIII

Perform the following additions, and explain your work:

1.	**2.**	**3.**	**4.**	**5.**	**6.**
-2	-12	-8	9	14	-7
7	8	8	-3	-6	-3

7.	**8.**	**9.**	**10.**	**11.**	**12.**
-4	23	-11	-31	7	-9
-12	-15	-26	45	5	-5

13.	**14.**	**15.**	**16.**	**17.**	**18.**
-3	7	-6	11	-22	-72
4	5	-8	-4	31	65
-8	-9	-2	-9	15	-21

19. If a boy weighing 54 lb. were weighed while holding a toy balloon which pulls upward with a force of 6 lb., what would be the *combined* weight? If $+54$ lb. represents the weight of the boy, what would represent the *weight* of the balloon?

20. In Ex. 19 the *combined* weight of the boy and the balloon is $(+54) + (-6)$ lb.; hence *adding* the negative number destroys part of the positive number; is this true in general for additions of positive and negative numbers? Illustrate your answer.

21. When does the addition of a negative number to a positive number destroy the latter wholly? When only in part? Illustrate your answers by means of assets and liabilities (cf. Ex. 12, p. 20).

22. Ex. 21 suggests a useful rule for algebraic addition, viz.:

(1) To add two numbers with *like* signs, find the sum of their absolute (arithmetical) values, and to this prefix their common sign.

(2) To add two numbers with unlike signs, find the difference of their absolute values, and to this prefix the sign of the larger.

Test this rule in the examples on p. 21.

23. A wheelman after riding 37 miles westward from a certain point rides back 12 miles. If distances to the westward are indicated by positive numbers, show that $37 + (-12)$ miles indicates both his direction and his distance from the starting point.

24. Indicate by a sum of positive and negative numbers what temperature is now registered by a thermometer which stood at 54° above zero, then rose 2°, later fell 9°, and then rose $2\frac{1}{2}$°.

25. Find the following indicated algebraic sums: $18 + (-3) + (-10) + 2$; $42 + (-27) + (-64)$; $-5 + 18 + (-11) + 23$.

26. Is algebraic addition sometimes performed by arithmetical subtraction? Is it so when the two summands have like signs? when they have unlike signs? Illustrate your answers.

17. Subtraction of algebraic numbers. We already know how to subtract positive numbers (cf. §§ 5, 13); therefore we now need to consider only the question of subtracting negative numbers.

Now, since subtraction is the inverse of addition (§ 6) and since $7 + (-3) = 4$ (§ 16), therefore $4 - (-3) = 7$; *i.e.*, $4 - (-3) = 4 + 3$. Similarly: $11 - (-5) = 11 + 5$; $-3 - (-12) = -3 + 12$; etc.

Hence, subtracting a *negative* number from any given number gives the same result as *adding its opposite* to the given number.

EXERCISE IX

Subtract the numbers written below, each from the one above it, giving the necessary explanation in each case:

1.	**2.**	**3.**	**4.**	**5.**
12	4	-2	8	-4
8	7	5	-3	-2

6.	**7.**	**8.**	**9.**	**10.**
-3	11	6	-4	-31
-10	-8	-15	-12	-25

11. The weight of a boy while holding a toy balloon, which pulls upward with a force of 6 lb., is 48 lb. If we take away (subtract) the balloon, how much will the boy weigh? Show that in this case $48 - (-6) = 54$.

Supply the missing numbers in the following equations:

12. $\because\ -5 + ? = -2, \ \therefore\ -2 - (-5) = ?$

13. $\because\ -5 + ? = -9, \ \therefore\ -9 - (-5) = ?$

14. $\frac{1}{2} - (-\frac{1}{3}) = ?$

15. $-1\frac{1}{8} - (-5\frac{3}{4}) = ?$

16. $10 - 3 - (-5) = ?$

17. $23 - (-a) + (-3) = ?$

If $a = 5$, $b = -6$, $c = -3$, and $d = 2$, find the value of each of the following algebraic sums:

18. $a - b + c + d.$

19. $a + c - (b - d).$

20. $a - (b + c - d).$

21. $a^2 + b - (d^2 - c).$

22. $ad - b + (a^2 - b).$

23. $2a^3 - c - \overline{3d^4 + b}.$

24. Using positive numbers to represent assets, illustrate the fact that subtracting a negative from a positive number increases the latter (cf. Ex. 11).

25. Make up concrete examples (like Ex. 11 or Ex. 24) to illustrate Exs. 4, 8, and 3, above.

26. Solve Exs. 1, 2, 5, and 9, above, by *counting*; and explain in each case *why* you count in one direction rather than in the opposite direction.

27. A rule for subtraction is often stated thus: "Reverse the sign of the subtrahend and proceed as in addition." Show that this rule is correct when the subtrahend is a negative number.

28. Mt. Washington is 6290 feet above sea level, Pikes Peak is 14,083 feet above sea level, and a place near Haarlem, in Holland, is $16\frac{1}{2}$ feet below sea level. By subtraction find how much higher Pikes Peak is than Mt. Washington; and also how much higher Mt. Washington is than the place near Haarlem.

29. When is algebraic subtraction equivalent to arithmetical addition (cf. Ex. 26, p. 22)? Illustrate your answer.

30. Write the following algebraic sums so that they shall not contain minus as a sign of operation, then find the value of each: $36-19-13+2$; $14\,a-26\frac{1}{2}\,a+9\frac{1}{4}\,a-15\,a$; $-6\,x-47\,x-3\,x$.

18. Product of algebraic numbers. Rule of signs.* The product of any two algebraic numbers is the result obtained by performing upon the multiplicand the same operation as that which is performed upon positive unity to obtain the multiplier [§ 7 (ii)].

E.g., since $3=1+1+1$,

therefore $8\cdot 3=8+8+8=24$,

the product 24 being obtained from 8 just as 3 is obtained from positive unity.

Similarly, $-8\cdot 3=(-8)+(-8)+(-8)=-24$. [§ 16

Again, since $-3=-1-1-1$, *i.e.*, since -3 is obtained by subtracting positive unity three times,

therefore $8\cdot(-3)=-8-8-8=-24$,

and $-8\cdot(-3)=-(-8)-(-8)-(-8)=8+8+8=24$.

Observe that two of the above products are positive and two are negative. How do the signs of the factors compare when the product is positive? when it is negative? How does the absolute value of the product (cf. § 15) compare with the absolute values of the factors?

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* Teachers who prefer to give more drill on addition and subtraction at this point may omit §§ 18 and 19 until after Chapter III has been read.

Applying the definition of a product to any two numbers whatever, just as we did to 8 and 3, -8 and 3, etc., above, we see that: (1) if two factors have *like* signs, their product is *positive;* (2) if they have *unlike* signs, their product is *negative;* and (3) the absolute value of the product equals the product of the absolute values of the factors.

EXERCISE X

Find the following products and explain your work:

1. $-5 \cdot 2$. **4.** $9 \cdot -3$. **7.** $3 \cdot -4 \cdot -5 \cdot 2$.

2. $-5 \cdot -2$. **5.** $-7\frac{2}{3} \cdot -6$. **8.** $2 \cdot -3x$, *i.e.*, $2 \cdot -3 \cdot x$.

3. $-12 \cdot 8$. **6.** $-n \cdot -5$. **9.** $-4a \cdot 3 \cdot -5 \cdot -7$.

10. Show that $(-2)^4$, *i.e.*, $-2 \cdot -2 \cdot -2 \cdot -2$, is 16. What is the value of $(-2)^5$? of $(-3)^2 \cdot (-4)^3$?

11. What is the sign of $(-7)^3$? Why? What is its absolute value? What is the sign of $(-7)^3 \cdot -2 \cdot 5$?

12. A succession of multiplications (as in Ex. 7, for instance) is called a **continued product.** Can the *sign* of a continued product be obtained without actually performing the multiplication? How? What is the sign if there are 5 negative factors?

13. An *odd* power of a negative number (*i.e.*, a power whose exponent is odd) has what sign? An *even* power? Is a power of a *positive* number ever negative? Explain.

If $a = -4$, $b = -2$, $c = 3$, $d = -1$, and $e = 2$, find the value of:

14. $abcde$. **16.** ab^3d^2. **18.** $b^2 - c^2 - d^2$.

15. $c^3d^2e^2$. **17.** $(a+b)^3$. **19.** $4c^2 - cd + d^2$.

Find the value of $(a+b) \cdot (x-y)$:

20. when $a = 2$, $b = -3$, $x = 3$, and $y = -5$.

21. when $a = -4$, $b = 6$, $x = 3$, and $y = -\frac{1}{2}$.

22. when $a = \frac{3}{4}$, $b = -2a$, $x = -\frac{1}{2}$, and $y = \frac{1}{4}$.

19. Division of algebraic numbers. Division is the inverse of multiplication (cf. § 8); *i.e.*, it consists in finding one of

two numbers when their product and the other number are given. Hence the results of § 18 may be used to show how to divide algebraic numbers.

Thus, since $8 \cdot 3 = 24$, $8 \cdot (-3) = -24$, $-8 \cdot 3 = -24$, and $(-8) \cdot (-3) = 24$,

therefore

$$24 \div 3 = 8,$$
$$-24 \div (-3) = 8,$$
$$-24 \div 3 = -8,$$
$$24 \div (-3) = -8.$$

So, too, whatever the given numbers. Therefore: (1) the absolute value of the quotient of two algebraic numbers is the quotient of their absolute values, (2) this quotient is *positive* if the dividend and the divisor have *like* signs, and (3) it is *negative* if they have *unlike* signs.

EXERCISE XI

Find the value of each of the following indicated quotients:

1. $14 \div 2$. **4.** $-3\frac{1}{4} \div (-1\frac{5}{8})$. **7.** $15 \div (-1)$.

2. $14 \div (-2)$. **5.** $-24 \div 9$. **8.** $-365 \div (-9\frac{1}{8})$.

3. $-18 \div 4\frac{1}{2}$. **6.** $(-6)^2 \div (-2)^3$. **9.** $-63a^2 \div (-9)$.

10. Of what operation is division the inverse? How, then, may the correctness of a quotient be tested? Illustrate.

11. If the dividend is positive, and the divisor negative, what is the sign of the quotient? Compare the signs of divisor and quotient when the dividend is positive; when it is negative.

Find the value of each of the following expressions:

12. $24 - 28 \div (-7) + (-16) \div (-4) \cdot (-3)$.

13. $-8 \cdot (-6) \div 24 - 27 \div (-6) \div 3$.

14. $\{28 \div (-7) - 2 \cdot (-4 - 2) + 24\} \div (-2)^3$.

Verify that $\dfrac{a+b}{x+y} \cdot \dfrac{a-b}{x-y} = \dfrac{a^2-b^2}{x^2-y^2}$:

15. when $a = 6$, $b = 2$, $x = 10$, and $y = 6$.

16. when $a = -8$, $b = 12$, $x = -9$, and $y = 7$.

CHAPTER III

ADDITION AND SUBTRACTION OF ALGEBRAIC EXPRESSIONS — PARENTHESES

I. ADDITION

20. Algebraic expressions, monomials, etc. Any combination of letters, or of letters and numerals, representing a number is called an **algebraic expression.**

The **terms** of an algebraic expression are the parts into which it is separated by the signs + and − (or, rather, these parts together with the signs preceding them). Thus, $3a$, $+m^2$, and $-5cx$ are the terms of the expression $3a+m^2-5cx$.

An algebraic expression which is not separated into parts by + or − signs is said to consist of a single term, and is called a **monomial.** Thus, $3a$, $5mx^2$, and $-11c^2x^2$ are monomials.

An expression consisting of two or more terms is called a **polynomial.** Thus, $3a+7m^2-5cx$ is a polynomial.

A polynomial consisting of two terms is usually called a **binomial**, and one of three terms, a **trinomial.** Thus, $2s-5xy$ is a binomial, while $4x-a+7k^2t$ is a trinomial.

21. Coefficients. Any one of the factors of a term, or the product of two or more of them, is called the **coefficient** (co-factor) of the product of the remaining factors. Thus, in the term $5axy^2$, the coefficient of axy^2 is 5, the coefficient of xy^2 is $5a$, the coefficient of $5xy^2$ is a.

A coefficient consisting of numerals only is called a **numerical coefficient**, while one that contains one or more letters is called a **literal coefficient.** Thus, in the term $-3ax^2m$, the

numerical coefficient of ax^2m is -3; but $-3a$ and $-3am$ are literal coefficients of x^2m and x^2, respectively.

Remark. The word "coefficient" is usually understood to mean "numerical coefficient" and to include the sign preceding the term. Observe also that ax means the same as $1ax$ (cf. § 9, Note).

22. Positive and negative terms. Like and unlike terms. A term preceded by the sign $+$ is called a **positive term**, and one preceded by the sign $-$ is called a **negative term.** If the first term of an algebraic expression is positive, its sign is usually omitted, but the sign of a negative term is *never* omitted.

Terms which either do not differ at all, or which differ only in their coefficients, are called **like terms**, also **similar terms**; terms which differ in other respects are called **unlike terms**, also **dissimilar terms.** Thus, $3x^2y$, $-5x^2y$, and $\frac{3}{4}x^2y$ are called similar terms.

EXERCISE XII

1. Name the coefficient of a^2x in each of the following terms:

$$3a^2x, \quad -5a^2x, \quad a^2x, \quad 4a^2bx, \quad -\tfrac{3}{8}a^2x, \quad \frac{12a^2bx}{7m}, \quad -9a^3x.$$

2. In Ex. 1 which coefficients are literal and which numerical? Which terms are positive and which negative?

3. Do the positive terms in Ex. 1 necessarily represent positive numbers for all values that may be assigned to the letters involved? Try $a=3$ and $x=-2$.

4. What is the coefficient of $x-y$ in each of the following expressions: $13(x-y)$, $-a(x-y)$, $\frac{5}{7}m(x-y)$, and $(4-a^3)(x-y)$? Which of these coefficients are numerical? Which literal?

5. Consult a good dictionary for the derivation of the words "monomial," "binomial," "trinomial," and "polynomial." Write three monomials, three binomials, three trinomials, and three polynomials.

6. Distinguish between the meanings of 5 in the expressions $5x$ and x^5. What name is given to the 5 in each case?

7. What are like terms? By what other name are they known? In what respects may they differ and still be like terms?

8. Are $3x^2y$, $-2x^2y$, and $\frac{5}{8}x^2y$ similar? Are $4ax^3$ and $-6bx^3$ similar? Are these last two terms similar if $4a$ and $-6b$ are regarded as their coefficients?

9. Write three sets of like terms, some terms being positive and some negative, and each set containing at least four terms.

23. Addition of monomials. Since 5 times any given number plus 2 times that number is 7 (*i.e.*, $5+2$) times the given number, therefore $5a + 2a = (5+2)a = 7a$, whatever the number represented by a. Similarly, $3mx^2y + 8mx^2y = (3+8)mx^2y = 11mx^2y$. Hence,

To add two or more similar monomials, add their coefficients and to this result annex the common literal factors, each with its proper exponent.

It is usually more convenient to write the terms to be added under one another, as in arithmetic, thus:

$$\begin{array}{r} 3xy \\ \underline{8xy} \\ 11xy \end{array} \qquad \begin{array}{r} 153a^2mx \\ \underline{74a^2mx} \\ ? \end{array} \qquad \begin{array}{r} 18aks \\ \underline{-7aks} \\ ? \end{array} \quad \text{(cf. § 16)}$$

If the monomials to be added are *dissimilar*, they cannot be united into a single term, but their sum may be *indicated* in the usual way; thus, the sum of $5a$ and $2x^2$ is $5a + 2x^2$. Similarly, the sum of $3m$ and $-6a$ is $3m + (-6a)$, which equals $3m - 6a$ (cf. § 16).

EXERCISE XIII

Add the following sets of similar terms, and explain your work:

1.	2.	3.	4.
$6n$	$18a^2$	$-9mx$	$31abx^2$
$3n$	$-10a^2$	$5mx$	$-22abx^2$
$-2n$	$-3a^2$	$-6mx$	$-6abx^2$

5. State a convenient rule for adding any number of like terms. Does your rule apply to cases in which some of the terms are negative?

Find the algebraic sum of:

6. $4x^2,\ -2x^2,\ -5x^2.$

7. $11ax,\ ax,\ -9ax.$

8. $-4cs^2,\ -cs^2,\ 8cs^2.$

9. $-3ax^ny,\ 5ax^ny,\ ax^ny.$

10. $12a^2n,\ a^2n,\ -4a^2n,\ -9a^2n.$

11. $3xz,\ -8xz,\ -xz,\ 2xz.$

12. $-ab^x,\ -7ab^x,\ ab^x,\ -5ab^x.$

13. $-xy,\ -4xy,\ 12xy,\ -3xy.$

Simplify the following expressions; *i.e.*, unite like terms, and indicate the results where the terms are unlike:

14. $3bxy^2 + (-4bxy^2) + (-12bxy^2) + 5bxy^2 + bxy^2 + (-bxy^2).$

15. $-4mp^3 + 13a^2x + 7mp^3 + 3mp^3 + (-5ax^2) - 2ax^2 + mp^3.$

16. $25c^3s^2 - 10b^2t - b^2t - c^3s^2 + 3b^2t + b^2t + c^3s^2 - 8c^3s^2.$

17. $7.5x + \frac{3}{2}x - x - \frac{1}{5}x + \frac{1}{6}x + \frac{1}{3}x - 3.45x + 1\frac{1}{2}x.$

18. $3d^4 - 5cg^3 + 2\frac{1}{3}cg^3 - 11.5d^4 - 7\frac{1}{2}cg^3 - 5d^4 + d^4 - cg^3.$

19. $-6(a-b^2) + 3(a-b^2) - (a-b^2) - 5(a-b^2) + (a-b^2).$

20. $23a^2 + 5b^n - 8a^2b^n - 13b^n + 24a^2b^n - 19a^2 + a^2 - b^n - a^2b^n.$

21. How many x's in $5x + 3x$? in $10x - 2x$? in $8x - ax$? in $4x - x$? in $x - 5x + 11x$? in $mx + nx - 3x$?

22. How many s^2t's in $8s^2t + 2s^2t$? in $3s^2t + as^2t$? in $3^2s^2t + 2ms^2t - s^2t$? in $3as^2t + (-2bs^2t)$?

23. Add $15x^2,\ -2ax^2,\ -7x^2,\ 4bx^2$, and $-2lx^2$.

[In Exs. 23–25 let a, b, and l belong to the coefficients.]

24. $2axy,\ -8xy$, and $3lxy$.

25. $5xz^2,\ -axz^2$, and $2bxz^2$.

24. Addition of polynomials. Any two polynomials, *e.g.*, $3a^2 - 7xy + 12y^2$ and $5a^2 + 6xy - 3y^2$, may be added thus:

$$\begin{array}{r} 3a^2 - 7xy + 12y^2 \\ 5a^2 + 6xy - \ \ 3y^2 \\ \hline 8a^2 - \ \ xy + \ \ 9y^2 \end{array}$$

This procedure may be stated thus: *To add two or more polynomials, write them under one another so that similar terms shall stand in the same column, and then add each column separately as in § 23.*

EXERCISE XIV

Add the following sets of polynomials:

1.
$$\begin{array}{r} 4a-2b \\ 2a+5b \\ \hline \end{array}$$

2.
$$\begin{array}{r} 6m+3n^2 \\ 4m-7n^2 \\ \hline \end{array}$$

3.
$$\begin{array}{r} ax-5y \\ 2ax-y \\ \hline \end{array}$$

4.
$$\begin{array}{r} 8p+2s-3t^2 \\ 11p-5s+7t^2 \\ \hline \end{array}$$

5.
$$\begin{array}{r} 6m-4n+7p \\ -2m+n-5p \\ -8m+2n-4p \\ \hline \end{array}$$

6.
$$\begin{array}{r} 3a-4x^2+5y \\ 6a+5x^2-y \\ -4a+2x^2 \\ \hline \end{array}$$

7.
$$\begin{array}{r} -a^2x-8b+2y^r \\ -2a^2x-5y^r \\ 8a^2x+2b-9y^r \\ \hline \end{array}$$

8. $12ax-5x^2-9y$, $-3ax-6x^2+2y$, and $ax+x^2-y$.

9. $3m-7n+2p$, $m+4n-6p$, and $n-2m+p$.

10. x^2-2x+1, $x-3+8x^2$, and $4x-3$.

11. $2a-7b+3(x^2-1)$, $4b-a-6(x^2-1)$, and $b+(x^2-1)$.

12. $3(a+b)-2c^t-5$, $7-6(a+b)+4c^t$, and $c^t-(a+b)$.

13. $2(m-n)+4(p+1)$, $3a-5(m-n)$, and $-7(p+1)-9a$.

14. $2\frac{3}{4}k-6.5l+3\frac{1}{2}m$, $5k-6\frac{1}{4}m$, and $4m+2l$.

15. $s^2-4\frac{1}{3}t-1\frac{1}{2}v$, $2t-3\frac{1}{6}s^2$, and $4\frac{2}{3}v-3\frac{1}{2}t+2s^2$.

Supply the missing coefficients in the following equations:

16. $12x-4ay-5x+7ay=\ ?\,x+?\,ay.$

17. $ax^2-2xy+dxy-cx^2=\ ?\,x^2+\ ?\,xy.$

18. $6rs-s^3+5cs^3+(2-3c)rs=\ ?\,rs+\ ?\,s^3.$

19. $(a^2-c)p+(3a+5c-5)p=\ ?\,p.$

20. Add $3x^2+4xy^2-y^n-7xy+2x^2y$, $10xy+(5c-10)xy^2$, $(c+1)x^2-3y^n$, and $xy+8xy^2+(2d-1)x^2y$.

25. Checking results. If a result (in addition, for example) is correct, then it must, of course, remain correct when we assign any arbitrary values to its letters. This is the basis

of a very useful test of the correctness of algebraic work (usually called a "check").

Thus, find the sum of $10x^2-3y$ and $-2x^2+9y$.

$$\begin{array}{r}
\text{SOLUTION} \\
10x^2-3y \\
-2x^2+9y \\
\hline
8x^2+6y
\end{array}
\qquad
\left.\begin{array}{r}
\text{CHECK} \\
=\ 10-\ 6=\ 4 \\
=-2+18=16 \\
\hline
=\ 8+12=20
\end{array}\right\}
\begin{array}{l}
\text{when } x=1 \\
\text{and } \quad y=2
\end{array}$$

And since the sum of 4 and 16, the *values* of the summands, is 20, which is also the *value* of the sum, therefore the work is *probably* correct.

EXERCISE XV

In Exs. 4–11, p. 31, check your results as above by putting any convenient arbitrary values for the letters.

II. SUBTRACTION

26. Subtraction of monomials. Since subtraction is the inverse of addition, therefore (cf. § 23):

To subtract one of two similar monomials from the other, subtract the coefficient of the subtrahend from that of the minuend, and to this remainder annex the common literal factors.

The work may be arranged thus:

$$\begin{array}{r}
73a^2 \\
24a^2 \\
\hline
49a^2
\end{array}
\qquad
\begin{array}{r}
27n^3k \\
-8n^3k \\
\hline
35n^3k
\end{array}
\qquad
\begin{array}{r}
36x^2y^3 \\
19x^2y^3 \\
\hline
?
\end{array}
\qquad
\begin{array}{r}
14ax \\
-9ax \\
\hline
?
\end{array}
\quad \text{(cf. § 17)}$$

If the given monomials are dissimilar, the subtraction can, of course, only be *indicated*.

A good practical rule for subtraction is: *To subtract one of two similar monomials from the other, reverse the sign of the subtrahend and proceed as in addition.* In order to avoid confusion in reviewing one's work, it is best, however, not actually to reverse the written sign but only to conceive it to be reversed.

EXERCISE XVI

Perform the following indicated subtractions:

1. $18-5$; $-18-5$; $-18-(-5)$; $18-(-5)$; $9-(-9)$.

2.	**3.**	**4.**	**5.**	**6.**
$7a$	$16bx^2$	$-18m^3$	$-18r^2x^3$	$26v^3y^5$
$4a$	$-3bx^2$	$7m^3$	$-7r^2x^3$	$-9v^3y^5$

7.	**8.**	**9.**	**10.**
$15cx^n$	$6m^2p^x$	$-34.7k^2x^3y^r$	$5\frac{2}{3}a^2m^4$
$3cx^n$	$-6m^2p^x$	$6.8k^2x^3y^r$	$-2\frac{1}{2}a^2m^4$

11. Show that "changing the sign of the subtrahend and proceeding as in addition" will give the remainder in each of the above exercises.

12. From $7ax^2y$ take $3ax^2y$; from $5np^3$ take $-8np^3$; from $4(a-2b^3)$ take $-11(a-2b^3)$; from the sum of $13y^2z^3$ and $-5y^2z^3$ take $4y^2z^3$.

13. Indicate the subtraction of b^2 from $3a^2$; of $4(c^2+y^2)$ from $-6(c+y)$; of $2xy$ from the sum of x^2 and y^2; of $-a^2b^n$ from the sum of $3a^4$ and $-b^{2n}$.

14. How many x^2y's in $8ax^2y-2x^2y$? in $mx^2y+nx^2y-2cx^2y$? in $7cx^2y-(-3x^2y)$?

15. Supply the missing coefficient in: $125mz-97mz=?z$; $c^2a^2-(-9a^2)=?a^2$; $5ax^2+?ax^2=2ax^2$; $4cs-?s=2bs$.

27. Subtraction of polynomials. One polynomial may be subtracted from another by writing the subtrahend under the minuend, similar terms under one another, and subtracting term by term, thus:

$$\begin{array}{r} 5b-3x^2+\ \ 6ab \\ 3b+5x^2-\ \ 9ab \\ \hline 2b-8x^2+15ab \end{array} \qquad \begin{array}{r} 3x^4-5\phantom{{}-2x} \\ -7x^4+4-2x \\ \hline 10x^4-9+2x \end{array}$$

EXERCISE XVII

In the following pairs of expressions subtract the second from the first, and check your results as in § 25:

1. $8a - 5b^2,\ 2a + b^2$.
2. $3m^2 - 7,\ m^2 - 10$.
3. $5x^2 + x,\ 3x - 2x^2$.
4. $s^2 + 3t,\ 2t - 5s^2$.
5. $a^2 - 2ab + b^2,\ -3a^2 + 12ab - 12b^2$.
6. $x^3 + 5x^2y + 7xy^2 - 2y^3,\ 3xy^2 - x^3 - 2y^3 - 5x^2y$.
7. Check your answer to Ex. 6 by letting $x = 2$ and $y = 1$.
8. From the definition of subtraction show that the minuend equals the sum of the subtrahend and remainder. What means of checking the result does this suggest (cf. § 17)?
9. In each of Exs. 1–6, p. 31, subtract the second expression from the first.
10. From $c^2 + d$ subtract $c^2 - d - 4k$. Check result in two ways.
11. From $5x^2 + 4a^2b$ take $8a^2b - 2x^2 + 5abx$, and check result.
12. Subtract $15y^6 + 10a^2x^3 + 4m^3y^n$ from $34a^2x^3 - 10m^3y^n$, and check result.
13. Subtract $15 - 3x + 10x^2$ from $12x^2 + 5$; also from $-2x$; also from 0.
14. Subtract $5\frac{1}{2}x^3 - 2\frac{1}{3} + x - 4\frac{2}{5}x^2$ from $7x^3 - 2\frac{1}{2}x + x^2 - 4$; also from 0.
15. Subtract the sum of $5a - 31b^2 + 2x^2$ and $26b^2 - 4x$ from $x^2 - 2a^2 + 7b^2$.
16. From what must $a^2b^n + 3cx + d^r$ be subtracted if the result is to be $4a^2b^n - d^r + 2cx$?

In Exs. 17–20 let a, b, and m belong to the coefficients:

17. From $2ax - 3by + mxy$ take $x - 2y + 5\frac{2}{3}xy$.
18. From $(m - 2b)y^2 + 3\frac{1}{2}z$ take $2z + (a - b)y^2$.
19. From $(b^2 - 1)xy + b^2y^3 - 3mx^3$ take $2b^2xy - 5mx^3 + ay^3$.
20. From $(a^2 - 3ab + m^2)x^2 + (4a^2 - 5ab + 2b^2 + 7m^2)y + 2amz^4$ subtract $(a^2 - 5ab + b^2 - m^2)x^2 - 3\frac{1}{2}a^2bz^4 + (a^2 - 2ab + m^2)y$.

III. PARENTHESES*

28. Parentheses removed and inserted. Such an expression as $2x-(y-3z)$ means that $y-3z$ is to be subtracted from $2x$; hence (§ 27)

$$2x-(y-3z)=2x-y+3z.$$

Similarly, $a-(-b+c-d-e)=a+b-c+d+e$; etc.

These equations (read from left to right) show that a parenthesis inclosing any number of terms, and preceded by the *minus* sign, may be removed *provided that the sign of each term within the parenthesis is reversed.*

Again, reading the above equations from right to left shows that any number of the terms of an expression may be inclosed within a parenthesis preceded by the minus sign, *provided that the sign of each term so inclosed is reversed.*

REMARK. A parenthesis preceded by the *plus* sign may, of course, be removed or inserted without changing the signs of the terms inclosed. (Why?)

EXERCISE XVIII

By means of § 27 show that:

1. $5a-(3a+b)=5a-3a-b=2a-b.$

[What is the sign of $3a$ in $(3a+b)$?]

2. $3x-2y-(-4x+y)=3x-2y+4x-y=7x-3y.$

3. $m^2-3np+p^2=m^2-(3np-p^2).$

4. $a-2b+c-4x=c-4x-(-a+2b).$

5. Using § 11, show that $8-(10-7)=8-3=5$; and then show that the same result may be obtained by using § 27, *i.e.*, show that $8-(10-7)=8-10+7=-2+7=5.$

Simplify each of the following expressions by two methods, as in Ex. 5, and compare results:

---

* "Parenthesis" here means any sign of aggregation whatever (cf. § 11).

6. $11-(3+6)$.

7. $11-(-3+6)$.

8. $-(8-5)+10$.

9. $27+(-5-3)-\overline{13-7}$.

10. $27-(-5-3)+\overline{13-7}$.

11. $-(6-4+9)+3-(-2+7)$.

12. In Ex. 9 what is the quality sign of 13? What does the minus sign preceding 13 indicate?

In Exs. 13–19 remove parentheses and unite similar terms:

13. $7x-3ac+(x-2ac)$. [Compare § 28, Remark.]

14. $3a-4b+(b-2a)$.

15. $2y^2-(-x^2+y^2-xy)$.

16. $5a^2+3b-(-2a)$.

17. $x-y+(x+y)-(3x-y)$.

18. $a-y^2-(a-3)-(-3y^2-1)$.

19. $-(2m-5)-(-6+x^2-3m)$.

In each of the following examples inclose the last two terms in a parenthesis preceded by $-$:

20. $2s-3t+w$.

21. $6+5x^2-3y$.

22. a^2+2b+c^4.

23. $ax^2-4bx-3+2y^3$.

24. $2h-3k-7x-5$.

25. $3m^k-2m^nx-5mx^n+x^k$.

29. Parentheses within parentheses. It often happens that one parenthesis incloses one or more others. In such cases the expression within an inner parenthesis forms *a single term* of the next outer parenthesis [cf. § 11 (ii)].

These parentheses, too, may be removed as in § 28, thus:

$$\begin{aligned}
&3a^2-\{9m-[-a^2-(4s^3-5m)+s^3]\} \\
&=3a^2-9m+[-a^2-(4s^3-5m)+s^3] && \text{[Removing brace} \\
&=3a^2-9m-a^2-(4s^3-5m)+s^3 && \text{[Removing bracket} \\
&=3a^2-9m-a^2-4s^3+5m+s^3 && \text{[Removing parenthesis} \\
&=2a^2-4m-3s^3. && \text{[Collecting terms}
\end{aligned}$$

Let the pupil simplify the above expression by first removing the *innermost* parenthesis, then the next innermost, and so on, and compare his work with what is here given.

EXERCISE XIX

Simplify the following expressions; that is, remove the parentheses and combine like terms:

1. $s-[t^2+(u^2-s)]$.

2. $s-[t^2-(u^2-s)]$.

3. $6a-[b-(-2a+3b)]$.

4. $x^2+[-y^2-(2y^2-3x^2)]$.

5. $(m-4p)-(a-\overline{p+m})$.

6. $3x^2-\{2a-(-x^2+a)\}$.

7. $-\{2a-(-x^2+a)\}$.

8. $-\{-(-x^2+a)\}$.

9. $mx^2-\{8y-(6x-mx)-2a\}$.

10. $-(60-25)-\{9^2-(18+27)\}$.

11. $3p+4q+[7p-2q-(5p-\overline{3-5q})]$.

12. $a-y-\{a-(-y-\overline{a-2})\}$.

13. $x-\{3x-[-(-3x+2y)+5y]-3y\}$.

14. $8x^2+2xy-[3x^2-y^2-(2xy-\overline{x^2+y^2})]-y^2$.

15. $2a^t-\{[3b^n-(a^t-\overline{2c-5a^t})]-[7b^n+5c-\overline{b^n-2a^t}]\}$.

16. $8a-2b-\{-(3c-d)-[4c-d-(-8a+2b)]-2d\}$.

17. $4-[5y-\{3-(2x-2)-4x\}]-\{x+5y-\overline{x+3}\}$.

18. In Ex. 2, how many minus signs affect u^2? How often, then, will its sign be reversed by removing the parentheses? What will be its sign finally? Answer the above questions for $-x^2$ in Ex. 7.

19. By considering the number of minus signs affecting the respective terms, remove *together* all parentheses in Ex. 9. Also in Exs. 11, 12, and 14.

20. In the expression $3m-4a+10x^2-5y+3ab^2-8ax$, inclose the 4th and 5th terms in a parenthesis preceded by the minus sign; then inclose this parenthesis, together with the two preceding terms, in a bracket preceded by the minus sign.*

21. Make the changes asked for in Ex. 20, in the expressions $3m+4a-10x^2-5y+3ab^2-8ax$, $3m-4a-2x^2+5y-3ab^2$, and $-5x^n+3y^m-4a-14bc+8k^2$.

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* The value of the expression is, of course, to be left unchanged.

CHAPTER IV

MULTIPLICATION AND DIVISION OF ALGEBRAIC EXPRESSIONS

I. MULTIPLICATION

30. Law of exponents in multiplication. What is the meaning of 5^2 (cf. § 9)? of a^3? of x^n?

How many times is s used as a factor in the product $s^3 \cdot s^2$? Is $s^3 \cdot s^2$ equal to s^5? Explain why.

How may the exponent of the product $s^3 \cdot s^2$ be obtained from the exponents of the factors s^3 and s^2? Would your answer remain true if we were to put other exponents in place of 3 and 2?

Is $x^4x^2x^5$ equal to x^{11}, *i.e.*, to x^{4+2+5}? Why? Is x^ax^b equal to x^{a+b}? Why?

The results of these considerations may be expressed in symbols thus:

$$\boldsymbol{a^m \cdot a^n \cdot a^p = a^{m+n+p}},$$

wherein a may stand for any number whatever, but m, n, and p are positive integers.

Translated into common language, this **law of exponents** is: *The product of two or more powers of any number is that power of the given number whose exponent is the sum of the exponents of the factors.*

31. Product of two or more monomials. The product of any two or more monomials may be obtained by a simple extension of § 30.

E.g., in the product of $2\,ax^2$ and $3\,ab^2x$, how many numerical factors? What is their product? How many a's in the

entire product? How many b's? How many x's? Write down this product, using the exponent notation. What is its sign? Why?

What is the product of $3\,a^2x^2$, $-2\,abx^2$, and $5\,ab$? Is it $-30\,a^4b^2x^4$? Explain in detail, mentioning the sign, the coefficient, the letters, and their exponents.

These considerations lead to the following rule for obtaining the product of two or more monomials: *To the product of the numerical coefficients of the several monomials, annex the different letters which these monomials contain, giving to each letter an exponent equal to the sum of the exponents of that letter in the several monomials.*

EXERCISE XX

Find the following indicated products, and explain your results, especially the signs and exponents:

1.	**2.**	**3.**	**4.**	**5.**
$3\,a^2$	$3\,a^2$	$6 \cdot 3^5$	$-8\,m^3s$	$-7\,a^2z^3$
$5\,a^4$	$-5\,a^4$	$-2 \cdot 3^2$	$-3\,ms^2$	$-2\,z^2$

6.	**7.**	**8.**	**9.**	**10.**
$-5 \cdot 2^3$	m^2x^2	$2\,x^n$	$-5\,a^2b^n$	$-4\,x^{2n}y^p$
$8 \cdot 2^7$	$-5\,a^3x$	$-6\,x^p$	$-8\,ab^2$	$-9\,x^ny^{3p}$

11. $7\,s^2 \cdot -3\,as \cdot -2\,a^2$.

12. $3\,mx^2 \cdot 2\,m^3 \cdot -7\,am$.

13. $-2\frac{1}{2}\,st \cdot -s^3 \cdot 3\frac{2}{5}\,a^2t$.

14. $-2\,a^nb \cdot 6\frac{3}{4}\,ab^3 \cdot 12\frac{2}{3}\,a^mb^4$.

15. $7.5\,mv^2 \cdot -4\frac{2}{3}\,am^3 \cdot -3\,a^nv$.

16. $-12\,k^2l^3 \cdot -akl^n \cdot 2\frac{1}{2}\,a^nk^4$.

17. $2\,(a-b)^2 \cdot -5\,(a-b) \cdot 3\,x^2y \cdot -2\frac{1}{5}\,x\,(a-b) \cdot 2.5\,(a-b)^3y^4$.

18. Define and illustrate the meaning of exponent, power, base (cf. § 9). May the base be negative? May it be a fraction? May the exponent be fractional or negative?

19. If n represents a negative number, is n^5 positive or negative (cf. Ex. 13, p. 25)? How does 3^4 compare with $(-3)^4$? 2^5 with $(-2)^5$?

20. What is the meaning of y^{n-2}? In this expression may n be less than 2? What is the product of $4\,a^3$ and $-7\,a^{n-2}$?

21. Determine, by inspection, the *sign* of the result in each of the following products when $a=-2$, $b=3$: $(a-b)^3$; $(a-b)^6$; $(a+b)^3$; $(ab^2)^5$; $(a^3b^2)^4$; $(a^3-b)^b$. State your reason in each case.

32. Product of a monomial and a polynomial. From the definition of a product [cf. § 7 (ii), § 18],

$$5\cdot(2+9)=5\cdot 2+5\cdot 9,$$

since $2+9$ is obtained by taking positive unity 2 times, then 9 times, and adding the two results.

Similarly, $\quad a(m+x-y)=am+ax-ay,$
whatever the numbers represented by a, m, x, and y.

Hence we may say: *To find the product of a polynomial and a monomial multiply each term of the polynomial by the monomial and add the partial products.*

The actual work may conveniently be arranged thus:

$$\begin{array}{l} 3a^2x-4x^2+11y^2 \\ \underline{-2xy\qquad\qquad\qquad\quad} \\ -6a^2x^2y+8x^3y-22xy^3 \end{array} \qquad \begin{array}{r} \text{CHECK} \\ \left.\begin{array}{r} =10 \\ \underline{=-2} \\ =-20 \end{array}\right\} \end{array} \begin{array}{l} \text{when } a=1,\ x=1, \\ \text{and } y=1 \end{array}$$

EXERCISE XXI

1. How is $2+a-x$ obtained from $+1$? How, then, may $y(2+a-x)$ be obtained from y?

	2.	**3.**	**4.**
Multiply	$3x-5y$	$a-4ab+3b^2$	$2m-3n^2-mn$
by	$2x$	$-5b$	$-4mn$

Find the following products, and check results (cf. § 25):

5. $(2x^3-4y^2)\cdot 7xy$.

6. $(4ax-5xy)(-2x^2)$.

7. $(-2s^3-3st)(-9st^2)$.

8. $(-6u^2+uv)(-4v^2x)$.

9. $-8x^3y^n(2x^m-4x^2y)$.

10. $5a^2(a^3-4a^2-2a+7)$.

11. $-7ax^2(8ax^2-3a^2x-5ax)$.

12. $-8a^2b^n(3-4a^m+12b^n)$.

13. $-12xy(2\frac{1}{2}x^2-5\frac{1}{3}x-4)$.

14. $2\frac{1}{2}abc(1.5a^3+7.5ab^2-6b^2)$.

15. $(3x^2z - 5x^3 + 4xz^2 - 9xz + 11)(-3xz)$.

16. $[x^3 + (a+1)x^2 + (a-1)x + 1] \cdot 2ax^2$.

17. $(xy^6 - 2x^2y^5 - 15x^4y^3 + 4x^5y^2 - 7x^6y)(-x^2y^{n-2})$.

18. $2(u^2 - 3v)[(u^2 - 3v)^5 - 13x(u^2 - 3v)^4 + 2x^2(u^2 - 3v) - 1]$.

19. $3a[7x^2 - 4(2x^2 + ax) + x(2a - 3x + 1)]$.

20. Multiply $3a - 5b + c - x - y$ by -1, and show that the result agrees with § 28.

21. By what must $x - 7y - 2az$ be multiplied, to obtain the product $12a^2x - 14a^2y - 4a^3z$?

22. Find a monomial and a polynomial whose product is $6ax^2 - 10a^2x - 14a^2x^2 + 8a^3x^2$.

23. Are the values of m and n in Exs. 9, 12, and 18, limited in any way? If so, how?

33. Product of two polynomials. Since $m + n$ is obtained by taking positive unity m times, then n times, and adding the two results, therefore (cf. § 32)

$$\begin{aligned}(a+b+c)\cdot(m+n) &= (a+b+c)m + (a+b+c)n \\ &= am + bm + cm + an + bn + cn.\end{aligned}$$

Similarly for any polynomials whatever; *i.e.*, *the product of two polynomials is obtained by multiplying each term of the multiplicand by each term of the multiplier, and adding the partial products.*

If any two or more terms of a product are similar, they should, of course, be united.

Such a multiplication and its check may be arranged thus:

$$\begin{array}{rl|l}
 & & \text{CHECK} \\
 & a^2 + 2ab - b^2 & = +2, \\
 & \underline{a + b\qquad\qquad} & \underline{= +2,} \quad \text{when } a = 1 \\
(a^2 + 2ab - b^2)\cdot a = & a^3 + 2a^2b - ab^2 & \qquad\quad \text{and } b = 1 \\
(a^2 + 2ab - b^2)\cdot b = & \underline{\qquad a^2b + 2ab^2 - b^3} & \\
 & a^3 + 3a^2b + ab^2 - b^3 & = +4,
\end{array}$$

Remark. The product of three or more polynomials may be obtained by multiplying the product of the first two by the third, this product by the fourth, and so on [cf. § 10 (1)].

EXERCISE XXII

Perform the following indicated multiplications:

1. $(x + 2a) \cdot (x - a)$.
2. $(3a^2 - 5x) \cdot (2a^2 + 3x)$.
3. $(7 - 2m^3) \cdot (3 - 5m^3)$.
4. $(a^2 + ab + b^2) \cdot (a - b)$.
5. $(2s - 3t^2) \cdot (3s - 4t)$.
6. $(x + a) \cdot (x + a)$, *i.e.*, $(x + a)^2$.
7. $(3m^2 - 10)^2$.
8. $(6x^2 - 3ay^2)^2$.
9. $(x^2 - xy + y^2) \cdot (x - y)$.
10. $(x^2 - xy + y^2) \cdot (x + y)$.
11. $(a^2 - 2ay + y^2) \cdot (a - y)$.
12. $(5s^3 - 2t^2) \cdot (5s^3 + 2t^2)$.
13. $(7e^2f^2 - 2g^3) \cdot (7e^2f^2 + 2g^3)$.
14. $(-4p^2q - 6r) \cdot (3p - qr^2)$.
15. $(m^2 - n^2 + p^2) \cdot (5m - 2np)$.
16. $(2x - 3y + z - 4)^2$.
17. $(a^2 - 3ab + b^2 - 2c)^2$.

18. $(3m^2 - 2mn) \cdot (5m + 3n^2) = ?$ Check your result by letting $m = 1$ and $n = 1$. If, in the product, the exponent of m should be wrong, would this check reveal the error? Explain. Would the error be revealed if m were taken equal to 2?

Multiply (and check your work):

19. $m^4 - 2m^3 - 6m^2 + m - 1$ by $3m^2 + m - 2$.
20. $2x^3 - 7xy + 3x^2 - 4x + 2y + 1$ by $xy - 3x - 2y$.
21. $a^2 - b^2 - c^2 - 2ab - 2bc + 2ac$ by $a - 2b + c$.
22. $1.8x^2 - 2xy - 2.3y^2 - 6x - 2.5y$ by $3\frac{2}{5}y - 1\frac{1}{2}x$.
23. $x^n + y^n$ by $x - y$; by $x^2 + y^2$; by $x^n - y^n$.
24. $x^m - 3x^{m-1}y + y^m - 3xy^{m-1}$ by $x^2 - 2xy + y^2$.
25. $2x + (3n - 1)y$ by $(n + 1)x - (3n + 1)y$.
26. $r^2 - (2rt - t^2)$ by $4t^2 - 4(2rt - r^2) - 1$.

34. Degree and arrangement of integral expressions. In multiplications with polynomials, and elsewhere, it is often advantageous to arrange the terms of a polynomial in a particular order; such arrangements will now be explained.

A term is said to be **integral** if it contains no *letters* in its denominator; it is *integral in a particular one of its letters* if that letter does not appear in its denominator. A polynomial

is integral, or integral in a particular letter, if each of its terms is so.

E.g., $3\,ax^2 + \dfrac{2\,bmy^2}{a} - \dfrac{5\,axy}{3}$ is integral in b, m, x, and y; it is fractional in a; its first and last terms are altogether integral, while its second term is integral only in b, m, and y.

By the **degree** of an integral term in any letter (or letters) is meant the number of times this term contains the given letter (or letters) as a factor. Thus, $7\,a^3x^2$ is of the second degree in x, and of the fifth degree in a and x together.

The degree of an integral polynomial is the same as the degree of its highest term. Thus, $3\,x^4 - 5\,a^3x^2y - 2\,bx^2y^3$ is of degree 4 in x, 3 in y, and 5 in x and y together.

A polynomial is said to be **arranged** according to ascending powers of some one of its letters if the exponents of that letter, in going from term to term toward the right, increase; and that letter is then called the **letter of arrangement.** If the exponents of the letter of arrangement decrease from term to term toward the right, the expression is said to be arranged according to descending powers of that letter.

Thus, $2\,x^3 - 5\,ax^2y - 7\,b^2xy^2 + 3\,m^2y^3$ is arranged according to descending powers of x, and ascending powers of y.

35. Multiplication in which the polynomials are arranged. If each of two polynomials is arranged according to powers of some letter which is contained in each, then their product will arrange itself according to powers of that letter, and the actual multiplication will take on an orderly appearance.

E.g., to find the product of $7\,x - 2\,x^2 + 5 + x^3$ by $3\,x + 4\,x^2 - 2$, arrange the work thus:

$$
\begin{array}{rrrrrrr|l}
 & & x^3 & -\ 2\,x^2 & +\ 7\,x & +\ 5 & & = 11, \\
 & & & 4\,x^2 & +\ 3\,x & -\ 2 & & = \ \ 5, \\
\hline
4\,x^5 & -\ 8\,x^4 & +\ 28\,x^3 & +\ 20\,x^2 & & & & \\
 & 3\,x^4 & -\ 6\,x^3 & +\ 21\,x^2 & +\ 15\,x & & & \text{when } x = 1 \\
 & & -\ 2\,x^3 & +\ 4\,x^2 & -\ 14\,x & -\ 10 & & \\
\hline
4\,x^5 & -\ 5\,x^4 & +\ 20\,x^3 & +\ 45\,x^2 & +\ x & -\ 10 & & = 55,
\end{array}
$$

(Check column heading: CHECK)

EXERCISE XXIII

1. Is $\frac{4}{3}\,a^2x^3$ integral or fractional? In what letters is $\dfrac{5\,a^2x\,y}{bc^2}$ integral? In what letters is it fractional?

2. May an integral term have a fractional coefficient? Illustrate. Write a term integral in m and n and fractional in d.

3. When is a polynomial fractional in a particular letter? Write a binomial fractional in a and b; a trinomial integral in a and b, but fractional in c.

4. In Ex. 11 below,

(1) Of what degree is each term of the multiplicand? each term of the multiplier?

(2) Of what degree in x is the first term of the multiplicand? Of what degree in y? Answer the same questions for the other terms of the multiplicand.

5. How is the degree of an integral polynomial determined? Give three illustrations from the exercises on p. 42.

6. Name the degree as regards x of each polynomial in Exs. 11 and 12 below.

7. Arrange the expression

$$3\,x^2y^5 + xy^6 - 8\,x^4y^3 - 6\,x^5y^2 + x^6y - 3\,y^7 + 5\,y^8$$

according to ascending powers of y. How is it then arranged with reference to x? Of what degree is this expression? What is the degree of $3\,x^2y^5$?

Multiply:

8. $6\,x^2 - 2 + 5\,x + 3\,x^3$ by $x^2 + 5 - x$.

[In Exs. 8–16 arrange both multiplier and multiplicand according to some letter contained in each, and observe that the product has then a corresponding arrangement.]

9. $2\,a + a^3 - a^2 - 1$ by $4 - a^2 + a$.

10. $3\,a^2x - 4\,ax^2 + x^3 - a^3$ by $a^2 - ax + x^2$.

11. $3\,xy^2 - y^3 - 3\,x^2y + x^3$ by $-2\,xy + x^2 + y^2$.

12. $x^2y^2 - xy^3 + y^4 - x^3y + x^4$ by $x^2 + xy - y^2$.

13. $4\,h^2r - hr^2 - h^3 + 2\,r^3$ by $h - 2\,r$.

14. $6y^4 + 6x^2y^2 + 2x^4 - 3x^3y - xy^3$ by $y^2 + 3x^2 - 2xy$.

15. $x^5 - 5x^3y^2 - 9y^5 - 6xy^4 + 15x^2y^3 + 2x^4y$ by $3y^2 + x^2 - 2xy$.

16. $4t^5 - 16s^2t^3 + 6s^5 + 8s^4t + 3st^4$ by $3s^3 - t^3 - 4s^2t$.

17. $a^{n+1} - 3a^{n+2} + a^{n+3} - 2a^{n+4}$ by $2a^{n-1} + 3a^{n-2} - 4a^{n-3}$.

18. $x^ny^3 + x^{n+2}y + x^{n+3} - x^{n+1}y^2$ by $x^3 + y^3 - xy^2 - x^2y$.

19. In Ex. 8, of what degree is the multiplicand? the multiplier? the product? The term of highest degree in the product is the product of what two terms?

20. Of what degree is the multiplicand in Ex. 11? the multiplier? What, then, should be the degree of the product? Should all the terms of the product be of the same degree? Why?

II. DIVISION

36. Law of exponents in division. Since division is the inverse of multiplication (§ 19), therefore the results of § 30 may be employed to find the law of exponents in division.

Thus: since $a^5 \cdot a^3 = a^8$, therefore $a^8 \div a^3 = ?$ Is $x^9 \div x^2$ equal to x^7, *i.e.*, to x^{9-2}? Why?

Write the following indicated quotients and explain your answer in each case: $a^7 \div a^4$; $s^5 \div s^2$; $2^7 \div 2^3$; $x^9 \div x$.

How is the exponent of the *product* of two powers of any given number obtained (cf. § 30)? How, then, should the exponent of the *quotient* be obtained?

If m and n are positive integers, m greater than n, and x any number whatever, then (cf. § 9, also Exs. 18, 20, p. 39) the above results may be expressed in symbols thus:

$$x^m \div x^n = x^{m-n}.$$

This equation states the **law of exponents** in division; translate this law into common language (cf. § 30).

37. Division of monomials. Since the quotient multiplied by the divisor always equals the dividend (§ 8), therefore $12x^5 \div 3x^2 = ?$ that is, what is the number which, when multiplied by $3x^2$, gives $12x^5$ as product (cf. § 31)?

Similarly: $8\,a^5x^8 \div 4\,a^2x^3 = ?$ Why? $24\,m^5y^4 \div 8\,m^3y^2 = ?$ Why? $-18\,a^4b^7 \div 6\,a^3b^2 = ?$ Why? $8\,m^4xy^3 \div (-4\,m^2xy) = ?$

How is the sign of the quotient determined? the coefficient? the exponents? How may § 8 be used to test the correctness of the quotient? From the above write a *rule* for dividing one monomial by another, mentioning the sign, coefficient, letters, and exponents of the quotient (cf. § 31).

EXERCISE XXIV

Perform the following divisions; check results by § 8:

1. $6\,a^3 \div 2\,a.$

2. $15\,a^4x^7 \div 3\,ax^2.$

3. $12\,m^2x^5 \div 4\,x^2.$

4. $-18\,a^3b^5 \div 6\,ab^3.$

5. $10\,c^6d^3e^4 \div 5\,c^3de.$

6. $-45\,m^5n^6 \div (-9\,m^4n).$

7. $\dfrac{-48\,a^7x^6}{12\,a^4x^5}.$

8. $\dfrac{15\,h^4p^3}{6\,hp^2}.$

9. $\dfrac{-50\,f^2g^4p^{10}}{25\,fg^2p^2}.$

10. $\dfrac{35\,a^4b^3}{-7\,a^4b}.$

11. $\dfrac{-\frac{2}{5}\,m^3z^9}{-\frac{1}{6}\,m^2z^7}.$

12. $\dfrac{-63\,mn^2p^3}{-9\,np^3}.$

13. $\dfrac{\frac{3}{4}\,f^2h^9}{-\frac{7}{12}\,f^2h^9}.$

14. $\dfrac{27\,m^{2a}n^{c+2}}{4\frac{1}{2}\,m^an^c}.$

15. $\dfrac{3.1\,(xyz)^5}{-.5\,x^2y^2z^2}.$

16. $\dfrac{-12\,a^{19}c^{17}}{3^2\,a^{14}c^{10}}.$

17. $\dfrac{2\,x^{m+3}}{6\,x^m}.$

18. $\dfrac{6\,x^{m+n}}{2\,x^n}.$

19. If two monomials have *like* signs, what is the sign of their product? of their quotient? How do we find the exponent of any given letter in the quotient of two monomials?

In Exs. 20–25, multiply the first monomial by the second; also divide the second monomial by the first:

20. $-16\,k^{19}l^6,\ 5\frac{1}{3}\,k^{19}l^{15}.$

21. $42\,(p+q)^3,\ -14\,(p+q)^5.$

22. $5\,x^py^n,\ 15\,x^{p+2}y^n.$

23. $\frac{1}{2}\,c\,(m+n),\ -10\,c^7\,(m+n)^2.$

24. $8\,ax^{3p}y^{2m+n},\ -2\,x^{3p+2}y^{2\,(m+n)}.$

25. $13\,(x-z)^3,\ -26\,(x-z)^4.$

38. Division of a polynomial by a monomial. Since (§ 32)

$$a(m+x-y) = am + ax - ay,$$

therefore $\quad (am + ax - ay) \div a = m + x - y,$

whatever the numbers represented by a, m, x and y. Hence,

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial, and add the quotients so obtained.

E.g., $(15\,a^2x^3 - 10\,bx^4y + c^2x^2) \div 5\,x^2 = 3\,a^2x - 2\,bx^2y + \frac{1}{5}\,c^2.$

EXERCISE XXV

Perform the following indicated divisions:

1. $\dfrac{4\,a^3 - 12\,a^5}{4\,a^2}.$

2. $\dfrac{-24\,x^2y^3 + 18\,x^3y^2}{6\,xy}.$

3. $\dfrac{c^7 - 5\,c^4 + 2\,c^3}{c^2}.$

4. $\dfrac{3\,m - 2\,n + 11\,x}{-1}.$

5. $\dfrac{9\,m^2n^3 + 12\,mn^2 - 30\,m^3n^4}{3\,mn^2}.$

6. $\dfrac{9\,m^2n^3 + 12\,mn^2 - 30\,m^3n^4}{-3\,mn^2}.$

7. $\dfrac{-18\,x^2 - 81\,x + 9\,x^5}{-3^2\,x}.$

8. $\dfrac{14\,r^2 - 21\,r^2s^3 + 8\,r^5}{7\,r}.$

9. $\dfrac{26\,a^3m^2 - 52\,a^2m^3 - 39\,a^4m^6}{13\,a^2m^2}.$

10. $\dfrac{7\,a^5b^3 - 4\,a^3b^7 + 6\,a^{10}b^4}{-2\,a^2b^3}.$

11. How may any polynomial whatever be divided by a monomial? How are the signs of the several quotient terms determined? their coefficients? their letters? their exponents?

Divide [and check the work in each case (cf. Ex. 10, p. 26)]:

12. $x + 4\,ax^2 - 3\,m^3x - 6\,amx$ by $-x$.

13. $a^5b^3m^3 - 4\,a^3b^7x^{10} + 12\,a^7b^4x$ by $4\,a^2b^3$.

14. $\frac{2}{3}\,r^2s + \frac{7}{8}\,cr^3s^6 - \frac{4}{3}\,r^4s^7$ by $2\,rs$; also by $\frac{4}{5}\,rs$.

15. $a^m - 2\,a^{m+1} - 5\,a^{m+2} + 9\,a^{m+4}$ by a^m; also by a^2.

16. $z^{n+4} - 3\,z^{n-1} + 4\,a^5z^7 - z^5$ by $-\frac{1}{2}\,z^3$.

17. $-10\,(h-1)^6 - 6\,(h-1)^5k + 15\,(h-1)^4k^2$ by $-5\,(h-1)$.

18. $x\,(x+y)^4 - x^2\,(x+y)^3 + x^7\,(x+y)^5$ by $-x\,(x+y)^2$.

19. $2\,(s-t)^m - s^2\,(s-t)^{m+1} - 5\,(s-t)^{m+3}$ by $\frac{1}{2}\,(s-t)^{m-1}$.

Separate each of the following expressions into two factors, one of which is x^2:

20. $c^2x^2 + d^2x^2.$

21. $a^4x^2 - a^2x^2 + x^2.$

22. $-3\,x^2y + 5\,x^2z - 7\,x^2.$

23. $-x^2 + 6\,e^5x^2 + \dfrac{e^3x^2}{4}.$

In Exs. 24–26, group the like powers of y (cf. Exs. 20–23):

24. $ty^2 + cy^3 - ry^2 - 3\,sy^2 + y^3$.

25. $ay^4 - 2\,by - 3\,cy^2 - my^4 + dy^2 - 9\,y$.

26. $(a+1)y^5 - (a-1)y^7 + y^7 - 3\,y^5 - (3\,a+4)y^3 + ay^7$.

39. Division of a polynomial by a polynomial. Since, by § 35, the product of $(4\,x^2 + 3\,x - 2)$ and $(x^3 - 2\,x^2 + 7\,x + 5)$ is $4\,x^5 - 5\,x^4 + 20\,x^3 + 45\,x^2 + x - 10$, therefore, with this last expression as dividend, and $x^3 - 2\,x^2 + 7\,x + 5$ as divisor, the quotient must be $4\,x^2 + 3\,x - 2$; *i.e.*,

$$(4\,x^5 - 5\,x^4 + 20\,x^3 + 45\,x^2 + x - 10) \div (x^3 - 2\,x^2 + 7\,x + 5)$$
$$= 4\,x^2 + 3\,x - 2.$$

The *process* of obtaining this quotient from the given dividend and divisor will now be explained.

Since the dividend is the product of the divisor by the quotient, therefore the *highest term* in the dividend is the product of the *highest term* in the divisor multiplied by the *highest term* in the quotient (cf. Ex. 19, p. 45); and therefore if $4\,x^5$, the highest term in the dividend, is divided by x^3, the highest term in the divisor, the result, $4\,x^2$, will be the highest term in the quotient.

Moreover, since the dividend is the algebraic *sum* of the several products obtained by multiplying the divisor by *each term* of the quotient, therefore if $4\,x^5 - 8\,x^4 + 28\,x^3 + 20\,x^2$, the product of the divisor by the highest term of the quotient, is subtracted from the dividend, the remainder, viz., $3\,x^4 - 8\,x^3 + 25\,x^2 + x - 10$, will be the sum of the products obtained when the divisor is multiplied by each of the other terms of the quotient except this one.

For the reason given above, if $3\,x^4$, the highest term of this remainder, is divided by x^3, the highest term of the divisor, the result, $3\,x$, is the next highest term of the quotient.

By continuing this process all the terms of the quotient may be found. The work may be arranged as follows:

$$
\begin{array}{rr|l}
 & \overbrace{4x^5-5x^4+20x^3+45x^2+x-10}^{\text{DIVIDEND}} & \overbrace{x^3-2x^2+7x+5}^{\text{DIVISOR}} \\
(x^3-2x^2+7x+5)\cdot 4x^2= & 4x^5-8x^4+28x^3+20x^2 & \underbrace{4x^2+3x-2}_{\text{QUOTIENT}} \\ \cline{2-2}
 & 3x^4-8x^3+25x^2+x-10 & \\
(x^3-2x^2+7x+5)\cdot 3x= & 3x^4-6x^3+21x^2+15x & \\ \cline{2-2}
 & -2x^3+4x^2-14x-10 & \\
(x^3-2x^2+7x+5)\cdot(-2)= & -2x^3+4x^2-14x-10 & \\ \cline{2-2}
 & 0 &
\end{array}
$$

CHECK

When $x=1$, dividend $=55$, divisor $=11$, and quotient $=5$, as it should.

Even if it is not known beforehand that the dividend is the product of two polynomials, the *process of division* may still be applied as above. This process may be formulated thus:

(1) *Arrange both dividend and divisor according to the descending powers of some one of the letters involved in each, and place the divisor at the right of the dividend.*

(2) *Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.*

(3) *Multiply the entire divisor by this first quotient term, and subtract the result from the dividend.*

(4) *Treat the remainder as a new dividend, arranging as before, and repeat this process until a zero remainder is reached, or until the remainder is of lower degree in the letter of arrangement than the divisor.*

EXERCISE XXVI

Divide (and check your results by § 25):

1. $x^2+7x+12$ by $x+3$.

2. x^2-x-20 by $x-5$.

3. $b^2-6b-16$ by $b+2$.

4. $s^2-14s+49$ by $s-7$.

5. $13x+6x^2+6$ by $3x+2$.

6. $8+3x^2-14x$ by $2-3x$.

7. $10a^4+11a^2-8$ by $1-2a^2$.

8. $3x^6-4x^3-7$ by $-x^3-1$.

9. $c^3+6c^2+12c+8$ by $c+2$.

10. $2x^3+11x^2+19x+10$ by $2x^2+7x+5$.

11. $75m^2+m^6-15m^4-125$ by $25+m^4-10m^2$.

12. $p^4+4p^3+6p^2+5p+2$ by p^2+p+1.

13. $2x^4+6x^2-4x-5x^3+1$ by x^2-x+1.

14. $3a^2+3a^4+3+3a+a^5+5a^3$ by $1+a$.

[Here, as in arithmetical "long division," labor may be saved by "bringing down" at any stage of the work only so much of the remainder as is needed for the next step.]

15. Divide $6a^2x^2-4a^3x-4ax^3+a^4+x^4$ by a^2+x^2-2ax.

SOLUTION

$$\begin{array}{r|l}
x^4-4ax^3+6a^2x^2-4a^3x+a^4 & x^2-2ax+a^2 \\
\underline{x^4-2ax^3+a^2x^2\qquad\qquad\quad} & \overline{x^2-2ax+a^2} \\
-2ax^3+5a^2x^2-4a^3x\quad & \\
\underline{-2ax^3+4a^2x^2-2a^3x\quad} & \\
a^2x^2-2a^3x+a^4 & \\
\underline{a^2x^2-2a^3x+a^4} & \\
0 &
\end{array}$$

NOTE. To make the explanation of § 39 apply when two or more letters are involved, replace "highest term" by "term of highest degree in the letter of arrangement."

16. In Ex. 15 perform the division when both dividend and divisor are arranged according to the descending powers of a.

17. Divide $4xy^2+8x^3+y^3+8x^2y$ by $y+2x$.

18. Divide $2a^4+k^4-5a^3k-4ak^3+6a^2k^2$ by k^2+a^2-ak.

19. $(10x^3y^2+x^5-10x^2y^3+5xy^4-5x^4y-y^5)\div(x^2+y^2-2xy)=?$

20. If the partial quotient, at any stage of the process of division, is multiplied by the divisor, and the corresponding remainder added, how must the result compare with the dividend?

21. What check for division is suggested by Ex. 20? Is this check more or less complete than that given in § 25? Explain.

22. Divide $2x^5+x^4+49x^2-13x-12$ by x^3-2x^2+7x+3.

[Since there is no term in x^3 in the dividend, care must be used to keep the remainders properly arranged.]

Divide (and check the results as the teacher directs):

23. $v^6-v^4-1+2v+v^3-v^2$ by $v-1+v^2$.

24. $a^5-41a-120$ by a^2+4a+5.

25. $m^4 + 16 + 4\,m^2$ by $2\,m + m^2 + 4$.

26. $\frac{9}{16}\,x^4 - \frac{7}{8}\,x^3y + \frac{19}{36}\,x^2y^2 + \frac{1}{6}\,xy^3$ by $\frac{3}{2}\,x + \frac{1}{3}\,y$.

27. $1.2\,ax^4 + a^3x^2 - 2\,a^2 - 3.4\,a^2x^3 + 6\,ax$ by $6\,ax - 2\,a^2$.

28. $(2\,x + 3\,x^3 - 1 + 2\,x^2)(1 + x^2 - x)$ by $1 + x + x^2$.

29. $a^5 - b^5$ by $(a^3 + b^3)(a + b) + a^2b^2$.

30. $a^3 + b^3 + c^3 - 3\,abc$ by $a^2 + b^2 + c^2 - ab - ac - bc$.

31. $x^4 - 3\,x^3 + x^2 + 2\,x - 1$ by $x^2 - x - 2$.

[In Ex. 31 the complete quotient is $x^2 - 2\,x + 1 + \dfrac{-x+1}{x^2 - x - 2}$.]

32. $v^2 - v + 7$ by $v + 4$.

33. $a^5 - 1$ by $a + 1$.

34. $2\,s^3 - 3\,s + 8$ by $s^2 - 4$.

35. $x^3 + x - 25$ by $x - 3$.

36. $a^4 - 7\,a^2 - 9\,a - 6\,a^3 - 6$ by $3 + a^2 - 2\,a$.

37. $3\,x^4 + 11\,x^3 + 11\,x^2 + 9\,x + 10$ by $4\,x + 5 + x^2$.

38. Divide $p^6 + q^6$ by $p + q$ until 4 quotient terms are obtained; divide 1 by $1 - r$ to 8 quotient terms; 1 by $1 - mx$ to 4 quotient terms; and a by $a - x$ to 5 quotient terms.

Divide:

39. $cd - d^2 + 2\,c^2$ by $c + d$.

40. $x^3 - y^3$ by $x - y$.

41. $a^4 - 16\,b^4$ by $a - 2\,b$.

42. $h^8 - k^8$ by $h^2 + k^2$.

43. $a^{2n} - x^{2n}$ by $a^n - x^n$.

44. $u^{2n} + 11\,u^n + 30$ by $u^n + 6$.

45. $x^{m+n} - x^ny^{n-1} - x^my^n + y^{2n-1}$ by $x^m - y^{n-1}$.

46. $x^{m+n-1} - 3\,x^ny^{2n-2} - 5\,x^{m-1}y + 15\,y^{2n-1}$ by $x^n - 5\,y$.

47. Divide $abc + ax^2 + x^3 + abx + bx^2 + cx^2 + acx + bcx$ by $x^2 + ax + ab + bx$.

Solution. Since x occurs in more terms than any other letter, it will be best to arrange the work thus (cf. Exs. 24–26, p. 48):

$$\begin{array}{l|l}
x^3 + (a + b + c)\,x^2 + (ab + ac + bc)x + abc & x^2 + (a + b)x + ab \\
\underline{x^3 + (a + b)x^2 \qquad\quad + abx} & \overline{x + c} \\
\qquad\qquad cx^2 + (ac + bc)x + abc & \\
\qquad\qquad \underline{cx^2 + (ac + bc)x + abc} & \\
\qquad\qquad\qquad\qquad\qquad 0 &
\end{array}$$

Divide:

48. $y^2 + cy + cd + dy$ by $y + c$.

49. $-ab + ay + y^2 - by$ by $y - b$.

50. $y^3 + 2\,dy^2 + 2\,y^2 + d^2y + 4\,dy + 2\,d^2$ by $y^2 + 2\,dy + d^2$.

51. $a - y^3\,(a + 6) + 5\,y^2 - y + 3\,ay^4 + a^2y^2 - 2\,ay$ by $3\,y^2 - y + a$.

52. Divide $x^2 - 21$ by $x - a$; note that the remainder is what the dividend would become if a were substituted for x.

53. Divide $x^2 + 3\,x + 1$ by $x - a$; note that the remainder differs from the dividend only in that a replaces x.

54. Divide $m^3 + 7$ by $m - c$ and compare the remainder with the dividend. Similarly, divide $v^3 - 1$ by $v - 2$; $5\,m^3 - 8\,m + 3$ by $m - 3$; $y^3 - 4\,y^2 + 3\,y - 1$ by $y - k$; $2\,r^4 - r^2 + 10$ by $r - 1$; by $r - 2$; by $r - 3$.

55. Divide $2\,xy^3 + 3\,x^4 - 4\,x^2y^2 - 7\,x^3y + y^4$ by $x^2 + y^2 - xy$, arranging first according to powers of x, then according to powers of y, and compare the results.

56. As has just been seen in Ex. 55, the *form* of the quotient depends upon the choice of the letter of arrangement *when the division is not exact;* is this the case when the division is exact?

40. Finite numbers. As we pass from left to right the numbers of the series 2, 2^2, 2^3, 2^4, etc., increase without end; and the numbers of the series 1, $\frac{1}{2}$, $\frac{1}{3}$, etc., decrease without end. Hence we see that, in mathematical operations, there *may* arise numbers which are greater, and others which are smaller, than any fixed number that we can name or even conceive of; such numbers are called **infinitely large** and **infinitely small** numbers, respectively. All other numbers are called **finite** numbers. An infinitely large number is represented by the symbol ∞.

41. Zero. Operations involving zero. (i) The result of subtracting any given finite number from itself is called *zero* (cf. § 13). Thus if a represents any finite number, then

$$a - a = 0.$$

(ii) From this definition of zero and the definitions of addition, subtraction, etc., already given, it follows that, if k is any finite number whatever,

then $$k + 0 = k - 0 = k,$$

and $$k \cdot 0 = 0 \cdot k = 0.$$

E.g., $k \cdot 0 = 0$ because $k \cdot 0 = k \cdot (a - a) = ka - ka = 0$.

(iii) If k is any finite number whatever, then

$$k \div 0 = \text{no finite number whatever.}$$

For, if $k \div 0 = f$, a finite number, then $f \cdot 0$ would equal k (§ 19), but this is impossible (ii).

(iv) $$0 \div 0 = f, \qquad \text{[any finite number}$$

for $$f \cdot 0 = 0.$$

(v) From (iii) and (iv) above it follows that *we must not divide by zero*, since doing so leads, at best, to an indeterminate result.

EXERCISE XXVII

1. When the values $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$ are assigned to x, how do the successive values of the fraction $5/x$ compare? Can you name a number so large that none of these values will exceed it? Can you name a number so near 0 that none of the series of numbers $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$ will be still nearer to 0?

2. What is meant by an infinitely small number? by an infinitely large number?

3. Define zero. How does it follow from your definition that

(1) $3 - 0 = 3$? (2) $0 \cdot 5 = 0$?

4. Can the equation $ax = 0$ be true if neither a nor x is zero? Does it require that both a and x should be zero?

5. What is the value of $\frac{0}{5}$? Why? What is the value of $0/a$, where a is any number except zero?

6. May $\frac{0}{0} = 5$? 1000? -72? $\frac{3}{11}$? Explain. What is meant by saying that $\frac{0}{0}$ gives an *indeterminate* quotient?

7. The quotient $\frac{5}{0}$ cannot be a finite number. Why? Will it be an infinitely large or an infinitely small number (cf. Ex. 1)?

42.* **Some elementary laws.** What is the meaning of the expression $5+2+8$ (cf. §§ 4 and 10)? of $2+5+8$? Wherein do these expressions differ?

(i) Although a change in the *order* in which operations are performed may, in general, change the result (cf. § 10), yet *some* such changes of order do not affect the result. Thus:

$$5+2+8=2+5+8=8+5+2,$$
$$5\cdot 2\cdot 8=2\cdot 5\cdot 8=8\cdot 5\cdot 2,$$
$$5+2+8=5+(2+8)=5+10,$$
$$5\cdot 2\cdot 8=5\cdot(2\cdot 8)=5\cdot 16,$$

and
$$5\cdot(2+8)=5\cdot 2+5\cdot 8.$$

(ii) Moreover, based upon our *experience* with particular sets of numbers, we have silently assumed, in the preceding pages, that the above changes may be made with any numbers whatever without affecting the result. Thus, if a, b, and c represent any numbers whatever (positive, negative, integral, etc.), we have assumed (without proof) that:

$$a+b+c=b+a+c=c+a+b, \text{ etc.,} \tag{1}$$
$$a\cdot b\cdot c=b\cdot a\cdot c=c\cdot a\cdot b, \text{ etc.,} \tag{2}$$
$$a+b+c=a+(b+c), \text{ etc.,} \tag{3}$$
$$a\cdot b\cdot c=a\cdot(b\cdot c), \text{ etc.,} \tag{4}$$

and
$$a\cdot(b+c)=ab+ac. \tag{5}$$

Of these equations, (1) states what is known as the *commutative law of addition;* (2), the *commutative law of multiplication;* (3), the *associative law of addition;* (4), the *associative law of multiplication;* and (5), the *distributive law of multiplication as to addition:* all of them taken together are often spoken of as the *combinatory laws of algebra.*

These laws are easily verified in any particular cases: throughout this book we shall continue to *assume* their correctness. We wish, however, to point out to the pupil that *mere verifications, however numerous, do not establish a general law.*

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* This article may, if the teacher prefers, be omitted till the subject is reviewed. For a full discussion of these laws see *El. Alg.* Chap. IV.

CHAPTER V

EQUATIONS AND PROBLEMS

43. Equation. Members of an equation. A statement that each of two expressions has the same value (*i.e.*, represents the same number) as the other, is called an **equation.** These two expressions are called the **members** of the equation, the expression preceding the sign of equality being the **first member,** and the other the **second member.**

Thus, $8x - 16 = 3x + 4$ is an equation; $8x - 16$ is its first member, and $3x + 4$ its second member.

REMARK. In algebraic work, the equation is a most important instrument; to it is due the chief advantage of algebra over arithmetic. We have already seen some evidence of this in § 3, but much more is to follow. In a recent book Sir Oliver Lodge says: "An equation is the most serious and important thing in mathematics."

44. Conditional equations. Identical equations. Is the statement $8x - 16 = 3x + 4$, true when $x = 1$? when $x = 2$? when $x = 3$? when $x = 4$? when $x = 5$? Answer the same questions with regard to $2x = (x + 1)^2 - (x^2 + 1)$.

The equation $3m + 5n = 22$ is true if $m = 4$ and $n = 2$, but is not true for any other positive integral values of m and n; while the equation $3x^2 + k - x^2 = k + 2x^2$ is true for *all* values that may be assigned to x and k.

An equation which is true for *all* values that may be assigned to its letters is called an **identical equation,** and also an **identity;** while one which is true only on condition that certain particular values be assigned to its letters, is called a **conditional equation.** In the following pages we shall use the word *equation* to mean *conditional equation* unless the contrary is expressly stated.

As we shall see later, by performing the indicated operations the two members of an identity may be reduced to exactly the same form; hence the name "identical equation."

45. Roots of an equation. Checking. The **roots** of an equation are those values which satisfy the equation; *i.e.*, they are those values which, when substituted for the letters the equation contains, make the two members identical.

Any process by which the roots of an equation are found is called **solving the equation.**

The final test of the correctness of supposed roots is to substitute them for the letters in the equation; if they satisfy the equation they are roots, otherwise not. This process is called **checking** the roots. Thus, 4 is a root of the equation

$$8x - 16 = 3x + 4,$$

because $8 \cdot 4 - 16 = 3 \cdot 4 + 4.$ [each member being 16

46. Some axioms and their uses. The following principles, usually called *axioms*, are useful in solving equations:

1. *If equals* (i.e., *equal numbers*) *are added to equals, the sums are equal.*

2. *If equals are subtracted from equals, the remainders are equal.*

3. *If equals are multiplied by equals, the products are equal.*

4. *If equals are divided by equals, the quotients are equal.* Here, however, as elsewhere, it is not permissible to divide by zero [cf. § 41 (v)].

The correctness of these axioms rests upon the fact that *equal* numbers are in reality the *same* number, differing at most in form. Thus, $24 + 11$, $7 \cdot 5$, and $6^2 - 1$ are merely different forms of writing 35.

SUGGESTION TO THE TEACHER. It is strongly recommended that the teacher illustrate the physical meaning of an equation, and also the meaning of the above axioms, by means of a pair of balances (easily made, if not provided by the school).

47. Solution of equations. To show how the above axioms may be used in solving equations, let it be required to solve the equation $8x - 16 = 3x + 4$, *i.e.*, to find the value of x which satisfies it.

SOLUTION

Since	$8x - 16 = 3x + 4,$	
therefore	$8x - 16 + 16 = 3x + 4 + 16,$	[Axiom 1
i.e.,	$8x = 3x + 20,$	
and therefore	$8x - 3x = 3x + 20 - 3x,$	[Ax. 2
i.e.,	$5x = 20,$	
whence	$x = 4.$	[Ax. 4

CHECK

On substituting 4 for x in the *original equation*, that equation becomes

$$8 \cdot 4 - 16 = 3 \cdot 4 + 4, \textit{ i.e.}, 16 = 16;$$

hence the equation is satisfied, and 4 is a root (cf. § 45).

EXERCISE XXVIII

1. Is 2 a root of $x^2 - 5x + 6 = 0$? Is 3 also a root? Explain. How may we check a supposed root of an equation?

Solve the following equations, give the reasons for each step in the work, and check the roots:

2. $10x = 40.$

3. $8y = -32.$

4. $k + 1 = 7.$

5. $m - 9 = 4.$

6. $2v + 7 = 63.$

7. $2v - 7 = 63.$

8. $46 = 5s - 4.$

9. $-13 = 3x + 8.$

10. $3m + 2 = m + 30.$

11. $7x - 10 = 5x + 18.$

12. $-4t = 3 + t - 15.$

13. $20 - 12u + 5 = 0.$

14. $13s - 9 - 2s = 24.$

15. $7x - 55 = 18 - 2x - 1.$

16. $6v - (v - 3) - 12 = 0.$

17. $y^2 - (y^2 + y + 8) = -6.$

18. In Exs. 7–13, point out the members of each equation. Which is the first member of the equation in Ex. 14? What is the other member called?

19. What is meant by solving an equation? Describe briefly the process used in solving an equation.

20. Are the equations in Exs. 2–17, above, conditional equations or identities? Why? In which class of equations would you place $2x+3=2(4x+3)-(6x+3)$? Why?

21. If $2a$ is subtracted from each member of the equation $5x+2a=3x+4b$, what is the resulting equation? What does this show with reference to removing a term from the first to the second member of an equation? Is the same thing true when a term is removed from the second member to the first? Show this by adding $-3x$ to each member of the given equation.

48. Transposition. Directions for solving equations. Removing a term from one member of an equation to the other is called **transposing** that term.

If	$x+a=b,$	
then	$x+a-a=b-a,$	[Ax. 2
i.e.,	$x=b-a.$	[$\because a-a=0$
Again, if	$x=b-a,$	
then	$x+a=b-a+a,$	[Ax. 1
i.e.,	$x+a=b.$	[$\because -a+a=0$

Hence, since a may represent any term whatever, *a term may be transposed from one member of an equation to the other by merely reversing its sign* (cf. also Ex. 21, above).

For solving equations such as those considered in § 47 the following directions may now be given:

1. *Transpose all the terms containing the unknown number to the first member of the equation, and all other terms to the second member.*

2. *Unite the terms of each member, and then divide both members by the coefficient of the unknown number.*

3. *Check the root thus found by substituting it in the given equation.*

Ex. 1. Solve the equation $4s - 15 = 2s + 11$.

SOLUTION

Transposing, we have $4s - 2s = 11 + 15$;
uniting like terms, $2s = 26$;
dividing by 2, $s = 13$.
Check: $4 \cdot 13 - 15 = 2 \cdot 13 + 11$. [each member is 37

EXERCISE XXIX

Solve (and check) the following equations:

2. $3y - 5 = 22$.

3. $-10 = 6a + 8$.

4. $3(x - 5) = 48$.

5. $7z + 2 - z = 17$.

6. $4 - x = -11 + 2x$.

7. $(d + 1)^2 - d^2 = -11$.

8. $20 - 5k = 3k + 3$.

9. $-8x = 4(x - 2) + 10$.

10. $4(-3 + h^2) = (2h - 3)^2$.

11. $\frac{2s}{3} - \frac{s}{4} = 10$.

[Multiply both members of the equation by 12 (see Ax. 3).]

12. $\frac{3k}{5} + \frac{k}{10} = 14$.

13. $v - \frac{v}{4} = 12$.

14. $2x + \frac{x}{3} = \frac{35}{6}$.

15. $\frac{m}{2} - \frac{3m}{4} = m + 10$.

16. $x^2 - (x^2 + 6x + 26) = 7x$.

17. $3y - 7 = 4 - 2y - 5$.

18. $\frac{1}{3}(y - 6) = \frac{1}{5}(y - 2)$.

19. $\frac{x+1}{7} + \frac{x+6}{2} = -2$.

20. $3\frac{1}{2}a = 5a - 9\frac{1}{2}a - 16$.

21 $12 - 3x + 20 = 44 + 3x$.

22. $\frac{x-9}{3} = \frac{x-5}{12} + 1$.

23. $14k - (20 - \overline{7k - 2}) = 6k + 68$.

24. $(c + 5)(2c - 1) - (2c - 3)(c + 7) = 0$.

25. What is meant by transposing a term from one member of an equation to the other? What change must be made in a term thus transposed?

26. State in order the axioms thus far used in solving equations. Illustrate the use of each. Why does the division axiom not apply when the divisor is zero? [Cf. § 41 (v).]

27. Point out the fallacy in the following reasoning:

If $x = a,$

then $x^2 = ax,$

and $x^2 - a^2 = ax - a^2,$ [subtracting a^2 from each member

i.e., $(x+a)(x-a) = a(x-a);$

therefore $2a(x-a) = a(x-a),$ [since $x = a$

and, therefore, $2 = 1.$ [dividing by $a(x-a)$

49. Translation of common language into algebraic language, and vice versa. The equation $x - 8 = 3$ is an *algebraic* sentence; it may be translated into *common* (verbal) language thus: "x exceeds 8 by 3" or "x is 3 greater than 8."

Similarly, the *verbal* statement "the excess of a over the product of s and t is 9," when expressed *algebraically*, becomes $a - st = 9$.

In order to use equations easily in the solution of problems we must learn to translate freely from either of these two languages into the other.

EXERCISE XXX

Write as algebraic sentences:

1. Nine is 2 greater than x.

2. y is 8 less than $3x$.

3. a^2 exceeds $2a$ by 1.

4. The excess of $8x$ over $6x$ is $2x$.

5. The difference of two given numbers is five less than three times their sum.

[Hint. Let a and b be the given numbers.]

6. The product of two given numbers exceeds half the larger number by 17.

7. Twenty-one is divided into two parts, the smaller of which is p. What is the larger part? Express by an equation that the larger part exceeds the smaller by 3.

8. Translate into verbal language the equations in Exs. 5–11, p. 57. In how many different ways may we translate the equation in Ex. 8?

9. A father is now 4 times as old as his son. Represent the age of each 5 years ago; 5 years hence. Also express by an equation the fact that 5 years ago the father's age was 7 times that of his son.

[HINT. Let the son's present age be s years.]

10. Translate into algebraic language the following statement: a rectangular flower bed whose length is y feet, and whose width is 6 feet less than its length, contains 40 square feet.

11. If butter costs m cents a pound, eggs n cents a dozen, and milk r cents a quart, express in algebraic language that

(1) the combined cost of 8 qt. of milk and 6 doz. eggs is $1.90.

(2) the cost of 9 qt. of milk is 30 cents less than the cost of $2\frac{1}{2}$ lb. of butter.

12. Express as common fractions: 50 % of n dollars; 26 % of k bushels; m % of $ 525. Show that the amount of x dollars at 5 % simple interest for 3 years is $x + \frac{3}{20}x$.

13. If the units' digit of a number is 2, the tens' digit 4, the number itself is $4 \cdot 10 + 2$, *i.e.*, 42. What is the number whose units' digit is 8 and whose tens' digit is 3? the number whose tens' digit is x and whose units' digit is $x + 7$?

14. The smallest of three consecutive integers is a; what are the other two? If n is any integer, $2n$ is an even integer; write the even integer next higher than $2n$; next lower than $2n$. Write the odd integer next lower than $2n$; next higher than $2n$.

15. A walks $2\frac{1}{2}$ miles an hour; B, 3 miles an hour. How far does each walk in 3 hours? in t hours? How much farther than A does B walk in 1 hour? Express by an equation that in $t + 2$ hours B walks 3 miles farther than A.

16. At the rate given in Ex. 15, in how many hours will A walk 10 miles? 15 miles? s miles? Answer the same questions for B.

17. If I can do a certain piece of work in 6 days, what part of it can I do in 1 day? in 5 days? in x days? If I can finish a job in d days, what part can I finish in 1 day? in 3 days?

50. Problems leading to equations. A **problem** is a question proposed for solution; it always asks to find one or more numbers which at the beginning are unknown, and it states certain relations (conditions) between these numbers, by means of which their values may be determined.

In solving a problem the important steps are:

1. *To represent one of the unknown numbers involved in the problem by some letter, as x.*

2. *To translate the common language of the problem into algebraic language.*

3. *To solve the equation thus found,—called the* **equation of the problem.**

4. *To check the result.*

These steps are illustrated in the solutions of the following problems.

Prob. 1. The sum of the ages of a father and son is 54 years, and the father is 24 years older than the son. How old is each?

SOLUTION. Stated in verbal language, the given conditions are:

(1) The number of years in the father's age plus the number of years in the son's age is 54.

(2) The number of years in the son's age plus 24 equals the number of years in the father's age.

To translate these conditions into algebraic language,

let x stand for the number of years in the son's age;

then, by the second condition,

$x + 24$ stands for the number of years in the father's age,

and, by the first condition,

$$x + x + 24 = 54,$$

which is the *equation of the problem.*

Solving this equation, we find $x = 15$, whence $x + 24 = 39$. On substitution in the problem, these numbers are found to satisfy

its conditions (*i.e.*, to check); therefore the father's and son's ages are, respectively, 39 years and 15 years.

Prob. 2. A boy was given 39 cents with which to buy 3-cent and 5-cent postage stamps, and was told to purchase 5 more of the former than of the latter. How many of each kind should he purchase?

SOLUTION. Stated in verbal language, the given conditions are:

(1) The total expenditure is 39 cents.

(2) There are to be 5 more 3-cent stamps than 5-cent stamps.

To translate these conditions into algebraic language,

let x stand for the number of 5-cent stamps purchased,

then $5x$ stands for the number of cents in their cost;

and, by the second condition,

$x+5$ stands for the number of 3-cent stamps purchased,

and $3x+15$ stands for the number of cents in their cost;

hence, by the first condition,

$$5x+3x+15=39,$$

which is the equation of the problem.

Solving this equation, we have $x=3$, whence $x+5=8$. Substitution in the problem shows that these values check. Hence the number of 5-cent stamps is 3, and the number of 3-cent stamps is 8.

Prob. 3. If a certain number is diminished by 6, and twice this difference is added to 5 times the number, the result will equal 88 minus 3 times the number. What is the number?

SOLUTION. To form the equation of the problem,

let n represent the number sought,

then $5n=5$ times the number,

and $2(n-6)=$ twice the difference of this number and 6,

and $88-3n=88$ minus 3 times the number.

Hence the given condition becomes

$$5n+2(n-6)=88-3n.$$

The solution of the equation gives $n=10$, which checks; therefore 10 is the required number.

Prob. 4. A number consists of two digits whose sum is 5; if the digits are interchanged, the number is diminished by 9. What is the number?

Solution. Let x represent the digit in the units' place; then, by the first condition,

$5-x=$ the digit in the tens' place,

and $10(5-x)+x=$ the number, [cf. Ex. 13, p. 61.

and $10x+(5-x)=$ the number with its digits interchanged.

Hence, by the second condition,

$$10x+5-x=10(5-x)+x-9,$$

whence $x=2$, and $5-x=3$.

These two digits are found to satisfy both conditions of the problem; therefore the number sought is 32.

EXERCISE XXXI

5. John has 14 cents less than Henry; together they have 60 cents. How much money has each?

6. Divide 28 into two parts whose difference is 4.

7. The sum of two numbers is 63, and the larger exceeds the smaller by 17. What are the numbers?

8. If 16 is added to a certain number, the result is the same as it would be if 7 times the number were subtracted from 56. What is the number?

9. Of four given numbers each exceeds the one below it by 3, and the sum of these numbers is 58. Find the numbers.

10. Divide \$2200 among A, B, and C in such a way that B shall have twice as much as A, and C \$200 more than B.

11. I take a trip of 90 miles, partly by train, partly by trolley. If I go 42 miles farther by train than by trolley, how far do I go by each?

12. Three boys together have 140 marbles. If the second has twice as many as the first, but only half as many as the third, how many marbles has each boy?

13. After taking 3 times a certain number from 11 times that number, and then adding 12 to the remainder, the result is less than 117 by 7 times the number. What is the number?

14. I spend $2.50 for 3-cent and 4-cent stamps, getting 25 more of the former than of the latter. How many of each kind do I buy?

15. A man who is 32 years old has a son who is 8 years old. How many years hence will the father be 3 times as old as his son (cf. Ex. 9, p. 61)?

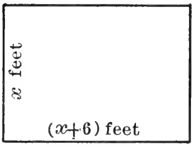
16. The sum of two consecutive integers is 73. What are the integers (cf. Ex. 14, p. 61)?

17. Find three consecutive integers whose sum is 51. Show that the sum of any three consecutive integers is 3 times the second of these integers.

18. The difference between the squares of two consecutive integers is 19. Find the integers.

19. Find two consecutive even integers whose sum is 98.

20. Find the even integer whose square subtracted from that of the next higher even integer leaves 52.



21. The accompanying diagram represents the floor of a room. If the perimeter (the distance around it) is 5 times the width, how wide is the floor? how long? How many square yards in its area?

22. The length and breadth of a rectangular floor differ by 5 ft.; the perimeter is 60 ft. Find the dimensions and area of this floor; also make an accurate diagram of the floor.

23. If each side of a square lot were increased by 2 yd., the area of the lot would be increased by 96 sq. yd. Find the side of the given lot. Draw an appropriate diagram.

24. A certain rectangle is 5 ft. longer than it is wide; if each dimension were increased by 2 ft., the area would be increased by 38 sq. ft. Find the length, the breadth, and the area of this rectangle.

25. Five boys agreed to purchase a pleasure boat, but one of them withdrew, and it was then found that each of the remaining boys had to pay $2 more than would have been necessary under the original plan. How much did the boat cost?

26. A laborer was engaged to do a certain piece of work on condition that he was to receive $2 for every day he worked, and to forfeit 50 cents for every day he was idle; at the end of 18 days he received $28.50. How many days did he work?

27. A number consists of two digits whose sum is 8; and if 36 is subtracted from this number, the order of its digits is reversed. What is the number?

28. In a certain two-digit number the tens' digit is twice the units' digit, and the number formed by interchanging the digits equals the given number diminished by 18. What is the number?

29. A two-digit number equals 7 times the sum of its digits; the tens' digit exceeds the units' digit by 3. Find the number.

30. What principal at 5% interest yields an annual income of $250? What principal at 4% simple interest amounts in 5 years to $2400 (cf. Ex. 12, p. 61)?

31. I make two equal investments, one at 6%, one at 4%. If the difference in the annual income from the two is $80, find the total sum invested.

32. Two trains which travel, respectively, 30 and 50 miles an hour, start toward each other at the same time from two cities 240 miles apart. How long before they meet?

Suggestion. Let $x =$ the number of hours before they meet; then $30x + 50x = 240$ (cf. Ex. 15, p. 61).

33. Two bicyclists ride toward each other from towns 104 miles apart, the first at the rate of 12 miles an hour, the second at the rate of 14 miles an hour. If they start at the same time, how long before they meet (cf. Ex. 15, p. 61)?

34. Two bicyclists, A and B, whose rates are, respectively, 12 and 15 miles an hour, start from the same town and ride in the same direction. If A starts $1\frac{1}{2}$ hours before B, how long before B overtakes him?

35. A walks m miles at the rate of 3 miles an hour, returning at the rate of $2\frac{1}{2}$ miles an hour. If the entire walk is made in $5\frac{1}{2}$ hours, what is the value of m?

HINT. The first half of the walk can be made in $\frac{m}{3}$ hours, the second in $\frac{2m}{5}$ hours (cf. Ex. 16, p. 61).

36. A tourist climbs a certain mountain at an average rate of 2 miles an hour, and descends at an average rate of 3 miles an hour. If the round trip takes 6 hours, how long is the path?

37. The difference of the radii of two circles is 4 inches; the sum of their circumferences is 88 inches. Find the radius of each. (The circumference of a circle equals 2π times its radius; and $\pi = 3\frac{1}{7}$, approximately.)

38. Divide \$351 among three persons in such a way that for every dime the first receives, the second shall receive 25 cents, and the third a dollar.

39. Divide 48 into two parts such that twice the larger part equals 10 times the smaller part (cf. Ex. 34, p. 7).

40. Three times Harry's age equals 5 times the age of his sister; the sum of their ages is 24 years. How old is each?

41. A, working alone, can do a certain piece of work in 3 days; B, in 6 days. In how many days can they complete it, working together (cf. Ex. 17, p. 62)?

HINT. Let $x =$ the required number of days; then $\frac{1}{x} = \frac{1}{3} + \frac{1}{6}$.

42. Solve Prob. 41, if A can do the work in 8 days, B in 6 days; also, if A can do the work in $4\frac{1}{2}$ days, B in 4 days.

REVIEW EXERCISE—CHAPTERS I–V

1. Define: negative number, absolute value of a number, coefficient, exponent, term, polynomial, degree of a term, finite number, equation, root of an equation, identity. Illustrate each of your definitions.

2. Define and illustrate: inverse operations, multiplication, division. Point out at least one advantage which the definition of multiplication given in § 7 (ii) has over that in § 7 (i).

3. If distances above sea level are called positive, what would -25 feet mean? $+35$ feet? What is the difference between these elevations?

4. A walks east at the rate of 3 miles an hour, B at the rate of -2 miles an hour. How long before the two are 15 miles apart? Illustrate by a drawing.

5. Translate into algebraic language:

(1) The number formed by interchanging the digits of a certain two-place number exceeds the number itself by 18.

(2) A certain number diminished by 5% of itself equals 76.

(3) The sum of two consecutive even integers equals half the difference of their squares.

6. Point out at least one advantage in using letters to represent numbers.

7. How are two or more similar monomials added? State a rule for subtracting one polynomial from another.

8. How may a parenthesis which incloses several terms, and which is preceded by the minus sign, be removed without affecting the value of the expression? Why?

9. State the law of signs for multiplication; for division. What powers of -3 between the 1st and 12th are negative in sign? Why? Find the value of $(-\frac{2}{3})^3 \cdot (-10)^2 \div (-\frac{5}{6})^4$.

10. State the exponent law for multiplication; for division. By reference to these laws find the value of $\dfrac{3\,a^{m+1}b^2 \cdot 5\,a^{m+3}b^n}{a^2b^{n+1}}$.

If $c = 9$, $d = -5$, $e = -2$, $f = -\frac{2}{3}$, $g = 1$, find the value of:

11. $5\,c + 3\,d - e + f - c \div f + 2\,c \cdot 3\,g \div e + 8\,f$.

12. $4\,c^2f^2 \div 6\,eg^3 + d^3 + 5[e - 3\,f + c^2 - 3\,ed]$.

13. $-\frac{1}{14}[\frac{1}{3}\,c - (12\,g^2 - 6\,efg)] + (c^2 + d^2) \div (c + d)^2$.

Perform the following indicated operations:

14. $(x - 5\,y)(4\,y - x)$.

15. $(am - cn)(3\,a + 5\,c)$.

16. $(k - 2\,q)^3$.

17. $(a^x - 1)^2$.

18. $(m^2 - mn + r)(m^2 + mn - r)$.

19. $(b^{2x} + c^x)(b^{2x} - c^x)$.

20. $(x^{2a} + y^{2c-1})(x - 1)$.

21. $(r^{2m} - r^m + 1)(3 + r^{m-1})$.

22. $(x^{2c+4} - 5x^{c+2} + 6) \div (x^{c+2} - 3)$.

23. $(x^{4c} - y^{4d}) \div (y^d - x^c)$.

24. $(\frac{1}{2}x^2 + \frac{1}{3}x - \frac{1}{6}) + (\frac{5}{8}x - \frac{3}{4}x^2 - x^3) - (-2x^2 + \frac{3}{5}x^3 - \frac{1}{8})$.

25. $4p - \{p^2 + 2pr - (2q + p^2)\} + (1 - pr + q)$.

26. $5m - 3n - \{-7n + \overline{m - 5n} - 3m\}$.

27. $2x - \overline{x - y + 2z} - [-\{-(y + 4z)\}]$.

28. $a(b + c - d) - b(a - 2c + d) - 3c(-a - d)$.

If $M = 7ab - 3b^2 - 4a^2$, $N = 3b^2 - 4a^2 - ab$, $P = a - b$, and $Q = b^4 - 4a^3b - 4ab^3 + 6a^2b^2 + a^4$, find the value of:

29. $M + N$.

30. $M - N$.

31. $\frac{2}{3}(4M + N)$.

32. $P^2 - 5N$.

33. M^2.

34. MP.

35. MN.

36. $N \div M$.

37. $Q \div P$.

38. $Q \div M$.

39. $2Q - NP$.

40. $Q \div P^3 + 3P^2$.

41. Check your work in Exs. 32–37. What two checks may be used for Ex. 30? for Ex. 36?

42. In Ex. 44 below, insert the second and third terms in a parenthesis preceded by +; place the fourth and fifth terms under a vinculum preceded by − (see footnote, p. 37).

43. In Ex. 45 below, inclose the first three terms in a parenthesis preceded by −; place the last two under a vinculum preceded by +.

In each of Exs. 44–49, collect the coefficients of r, s, and t (cf. Exs. 24–26, p. 47).

44. $2a^2s - 3ar - bs - cs - 4r$.

45. $-r + 5f^2s + 2s + f^3r - 7ft$.

46. $-n^2t + 5c^2ds - 2b^2t - 4eft - 10c^2ds$.

47. $t + a^2r + b^2s + r + s - 2ct + 2bs + c^2t - 2ar$.

48. $(a + 2c)s + (c + d)r + 2cs - 5dr - 7ds - er$.

49. $(m^3 - 1)s - 2(1 - n^3)s + n^4r - 3s - (m^4 - 2m^2n^2)r + n^5t$.

50. Multiply $x^2 + (a + c)x$ by $x - c$; by $x^2 + ex + k$.

51. Divide $2x^3 + 2n^2x^2 - n^3x - mn^3 - x^2(2m + 2n) - nx^3 - 2mnx + mnx^2$ by $-x^2 + (m+n)x + mn$ (cf. Ex. 47, p. 51). Check your result by multiplication.

Solve, and check the roots:

52. $x(2x-5)-57=2x(x+7)$.

53. $\dfrac{x+2}{5}=\dfrac{2x+3}{3}+\dfrac{2-x}{2}$.

54. $\frac{1}{3}x-(\frac{5}{6}x+\frac{1}{7})=-x+\frac{2}{7}$.

55. $\frac{3}{4}(x-2)(x-1)=\dfrac{3x^2-48}{4}$.

56. Four times the number of seniors in a certain school exceeds the number of freshmen by 15. If the total enrollment in the two classes is 130, find the size of each class.

57. A certain number is subtracted from 50 and 42 in turn. If $\frac{1}{3}$ of the first remainder equals $\frac{1}{2}$ of the second, find the number.

58. If the population of a town has increased 30% in the last 10 years, and is now 5200, find the population 10 years ago.

59. A man spends $\frac{1}{2}$ of his income for living expenses and insurance, $\frac{1}{24}$ for books, $\frac{1}{24}$ for travel, $\frac{1}{16}$ for charities. If he saves \$425, what is his income?

60. A certain rectangle is 8 ft. longer and 5 ft. narrower than a given square, and its area exceeds that of the square by 5 sq. ft. Find the side of the square. Draw an appropriate figure.

61. A and B together can do a certain piece of work in 6 days. If A can do it alone in 10 days, in how many days can B do it?

62. Silk marked to sell at a gain of $33\frac{1}{3}$% has its marked price reduced 20%, and then sells for 80 cents a yard. Find its cost.

CHAPTER VI

TYPE FORMS IN MULTIPLICATION — FACTORING

I. SOME TYPE FORMS IN MULTIPLICATION

51. Type forms. Although all exercises in multiplication and division of integral polynomials can be readily solved by § 33 and § 39, yet there are a few special cases of these operations which occur so frequently in practice that it is well worth one's while to memorize them; they are often spoken of as **type forms**. Some of these type forms are considered in the next few paragraphs.

52. Square of a binomial. Let a and b represent any two numbers whatever; then by actual multiplication (§ 33),

$$(a+b)(a+b)=a^2+2\,ab+b^2,$$

and

$$(a-b)(a-b)=a^2-2\,ab+b^2;$$

i.e.,

$$(a+b)^2=a^2+2\,ab+b^2, \qquad \text{I}$$

and

$$(a-b)^2=a^2-2\,ab+b^2, \qquad \text{II}$$

whatever the numbers represented by a and b.

Translated into common language, I becomes:

The square of the sum of any two numbers equals the square of the first number, plus twice the product of the two numbers, plus the square of the second number.

The student may translate II into common language.

By means of I and II, we can now write down (without actually performing the multiplication) the expanded form of the square of any binomial whatever. Thus:

Ex. 1. $(m+3)^2=m^2+6\,m+9.$

Ex. 2. $(x-y)^2=x^2-2\,xy+y^2.$

Ex. 3. $(2\,s-3\,t)^2=(2\,s)^2-2(2\,s)(3\,t)+(3\,t)^2=4\,s^2-12\,st+9\,t^2.$

EXERCISE XXXII

Expand the following (check your work as teacher directs):

4. $(x+y)^2$.	**12.** $(1+z)^2$.	**20.** $(\frac{1}{2}a-2)^2$.
5. $(m+n)^2$.	**13.** $(7-v)^2$.	**21.** $(\frac{1}{4}y+6)^2$.
6. $(h+k)^2$.	**14.** $(2x-b)^2$.	**22.** $(5x^2-\frac{3}{4})^2$.
7. $(u+w)^2$.	**15.** $(a+7x)^2$.	**23.** $(\frac{5}{6}m+\frac{2}{3}n)^2$.
8. $(a-p)^2$.	**16.** $(3m^4-2)^2$.	**24.** $(2a^2x+3by^3)^2$.
9. $(c-h)^2$.	**17.** $(2g^2-5h)^2$.	**25.** $(5rs-3r^3s^2)^2$.
10. $(x+3)^2$.	**18.** $(11-7k)^2$.	**26.** $(x^n+y^n)^2$.
11. $(a-5)^2$.	**19.** $(4b^3+1)^2$.	**27.** $(3a^n-2s^m)^2$.

28. Expand: $(a+b-5)^2$, *i.e.*, $\{(a+b)-5\}^2$; $(c+2+d)^2$; $(-2c-d+e^3)^2$; $(7+a^2-c)^2$; $(3x^n-p-q)^2$.

29. Since $a-b=a+(-b)$, show that II, § 52, is included under I.

30. What must be added to x^2+6x to make it the square of $x+3$?

31. What must be added to $a^4+a^2b^2+b^4$ to make it the square of a^2+b^2?

32. What must be added to $25-10a^3$ to make it the square of $5-a^3$?

33. What must be added to $x^8+2x^4y^3+4y^6$ to make it the square of x^4+2y^3?

By the method of § 52 write down the squares of the following numbers:

34. 16, *i.e.*, $10+6$.	**36.** 28.	**38.** 71.
35. 19, *i.e.*, $20-1$.	**37.** 43.	**39.** 83.

40. Expand $(a+1)^2$; also $(-a-1)^2$. Compare and explain.

53. Product of sum and difference. If a and b represent any two numbers whatever, then, by actual multiplication,

$$(a+b)(a-b)=a^2-b^2,$$

whatever the numbers represented by a and b.

The student may translate this formula into common language (cf. § 52).

EXERCISE XXXIII

Write the following products by inspection and check results:

1. $(x+y)(x-y)$.
2. $(m+n)(m-n)$.
3. $(3x+y)(3x-y)$.
4. $(x-2y)(x+2y)$.
5. $(4a+15b)(4a-15b)$.
6. $(6p-5q)(6p+5q)$.
7. $(2xy-7)(2xy+7)$.
8. $(4m^2-3n)(4m^2+3n)$.
9. $(9+5p^2r)(9-5p^2r)$.
10. $(x^3+\frac{1}{2}y^2)(x^3-\frac{1}{2}y^2)$.
11. $(10mn-6)(10mn+6)$.
12. $(\frac{4}{7}+a^3)(\frac{4}{7}-a^3)$.
13. $(y^m-11)(y^m+11)$.
14. $(ax^n+b^2)(ax^n-b^2)$.
15. $\{(x-y)+z\}\{(x-y)-z\}$.
16. $\{(a+b)+c\}\{(a+b)-c\}$.
17. $(m+n+p)(m+n-p)$.
18. $(c-d+5)(c-d-5)$.
19. $\{2-(x+y)\}\{2+(x+y)\}$.
20. $(7+m+n)(7-m-n)$.
21. $(a-b+x)(a+b-x)$.
22. $(2k-l+3)(2k+l-3)$.
23. $(9x^2-4y^2)\div(3x-2y)=?$ Why?
24. $(16a^2-25b^2)\div(4a+5b)=?$ Why?
25. $(x^4-y^4)\div(x^2-y^2)=?$ Why?
26. $(x^6-y^4)\div(x^3-y^2)=?$
27. $(x^{18}-y^8)\div(x^9+y^4)=?$
28. Find, by the above method, the product of 22 by 18, *i.e.*, of $20+2$ by $20-2$; of 17 by 23; of 42 by 38; of 56 by 44.

54. Product of binomials having a common term. By actual multiplication,

$$(x+3)(x+5)=x^2+8x+15=x^2+(3+5)x+15;$$

and $(x+3)(x-5)=x^2-2x-15=x^2+(3-5)x-15.$

And, in general,

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

whatever the numbers represented by a, b, and x.

Translating this formula into words, it becomes:

The product of two binomials having a term in common equals the square of the common term, plus the algebraic sum of the unlike terms multiplied by the common term, plus the product of the unlike terms.

EXERCISE XXXIV

Write down the following products (check as teacher directs):

1. $(a+5)(a+7)$.
2. $(a-5)(a-7)$.
3. $(a+5)(a-7)$.
4. $(a-5)(a+7)$.
5. $(y-c)(y+2c)$.
6. $(x^2+4)(x^2+5)$.
7. $(x^2+4)(x^2-5)$.
8. $(x^2-4)(x^2-5)$.
9. $(x^2-4)(x^2+5)$.
10. $(3+m)(5+m)$.
11. $(b+a)(c+a)$.
12. $(2x+1)(-5+2x)$.
13. $(a-b)(a-c)$.
14. $(4+3a)(-6+3a)$.
15. $(4s^2-5)(4s^2+1)$.
16. $(xy-4)(xy+16)$.
17. $(-8+m^2n^3)(2+m^2n^3)$.
18. $(s+r^2)(3s+r^2)$.
19. $\{(l+m)-2\}\{(l+m)-5\}$.
20. $\{(l+m)+8\}\{(l+m)-15\}$.
21. $(m-n-5)(m-n-9)$.
22. $(s-t+4)(s-t-4)$.
23. $(r^p-10)(r^p+15)$.
24. $(x^{2m}+3)(x^{2m}-7)$.
25. $(3a^5-x)(2x+3a^5)$.
26. $(b^2+y)(-c+y)$.
27. $\{3(ax)^2+2\}\{3(ax)^2+1\}$.
28. $(5-4x^2y^2)(5+x^2y^2)$.
29. $(m^2-c)(-7m^2-c)$.
30. $(3p-q-7)(3p-q+7)$.
31. When $b=a$, what does the formula of § 54 become? What does it become when $b=-a$? Are the formulas of §§ 52 and 53 only special cases of $(x+a)(x+b)=x^2+(a+b)x+ab$?

55. Product of two binomials whose corresponding terms are similar. By actual multiplication we obtain

$$\begin{array}{r}
3x+4y \\
5x-2y \\
\hline
15x^2+20xy \qquad \\
-\ 6xy-8y^2 \\
\hline
15x^2+14xy-8y^2
\end{array}$$

Here the term $14xy$ is the algebraic sum of the "cross products" $5x \cdot 4y$ and $-2y \cdot 3x$.

With a little practice the final product of two such binomials may be written down by inspection, *i.e.*, without first writing the partial products.

EXERCISE XXXV

Write the following products by inspection:

1. $(3x+2)(4x-3)$.
2. $(5m-1)(2m-3)$.
3. $(2r+5)(r-5)$.
4. $(a-11)(3a-1)$.
5. $(3x+2y)(4x+3y)$.
6. $(x-3y)(5x+6y)$.

7. In each of the above products, how is the first term obtained? the third? the second?

8. What is meant by the expression "cross products" as used in § 55? Illustrate from Ex. 3, above.

Write down the following products by inspection, and check results as the teacher directs:

9. $(7-2m)(7-m)$.
10. $(3-4a)(4+3a)$.
11. $(9x-2y)(x+y)$.
12. $(2a-4b^2)(5a-6b^2)$.
13. $(7c^2+d^2)(3c^2+8d^2)$.
14. $(a^m-2e)(a^m-e)$.
15. $(6x^p+4)(3x^p-2)$.
16. $(11-7cd^3)(6+3cd^3)$.
17. $(\frac{1}{2}c+2)(\frac{1}{3}c+1)$.
18. $(\frac{1}{3}a-\frac{2}{5})(\frac{3}{4}a+\frac{4}{5})$.
19. $(x+a)(x+b)$.
20. $(3x+c)(x+d)$.
21. $(3x-c)(x-d)$.
22. $(3x+c)(5x+d)$.
23. $(3x-c)(5x-d)$.
24. $(ky+1)(ny-1)$.
25. $(ky+a)(ny+b)$.
26. $(ky-a)(1-cy)$.

56. The square of any polynomial. By actual multiplication it is found that

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc,$$

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad + 2bc+2bd+2cd,$$

$$(a+b+c+d+e)^2 = a^2+b^2+c^2+d^2+e^2+2ab+2ac+2ad + 2ae+2bc+2bd+2be+2cd+2ce+2de,$$

and so on for any polynomials whatever; that is, *The square of any polynomial whatever equals the sum of the squares of all the terms of the polynomial, plus twice the product of each term by all the terms that follow it* (for proof see § 209).

EXERCISE XXXVI

Expand by inspection (check as teacher directs):

1. $(c+d+e)^2$.
2. $(m+n-s)^2$.
3. $(a-b-c)^2$.
4. $(m+r+1)^2$.
5. $(m-r-3)^2$.
6. $(2x+y+z)^2$.
7. $(2x+3y-z)^2$.
8. $(2x-3y+z)^2$.
9. $[-(2x+3y+1)]^2$.
10. $(3c^2+d-4)^2$.
11. $(4a^2+b^2+3c^2)^2$.
12. $(4a^3-b-5)^2$.
13. $(5+x-y^2)^2$.
14. $(-5-x+y^2)^2$.
15. $(a-b+c-d)^2$.
16. $(ax+by+cz)^2$.
17. $(\frac{1}{2}a-\frac{1}{3}c+\frac{1}{4}e)^2$.
18. $(mn-np-pq)^2$.
19. $(abx-acy-bcz)^2$.
20. $(2x-3y+4z-a)^2$.
21. $(x^m+x+1)^2$.
22. $(l+m+n+p+q+r+s)^2$.

23. Could any of the above products have been found by means of formulas already used (cf. § 52, also Ex. 28, p. 72)?

24. Give a rule for writing down the square of any polynomial whatever. What does this rule become when the polynomial is a binomial (cf. § 52)?

57. Cube of a binomial. The cube of a binomial is another product which, because of its frequent occurrence, should be memorized. By actual multiplication we obtain

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

and
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3,$$

whatever the numbers represented by a and b.

By means of these formulas (which the pupil should translate into words) we may write by inspection the cube of any binomial whatever.

NOTE. § 52 and § 57 are particular cases of what is known as the *binomial theorem*; this theorem is considered in § 112.

Ex. 1. $(x+2)^3 = x^3 + 3x^2\cdot 2 + 3x\cdot 2^2 + 2^3 = x^3 + 6x^2 + 12x + 8.$

Ex. 2. $(2a-5b)^3 = (2a)^3 - 3(2a)^2\cdot(5b) + 3(2a)\cdot(5b)^2 - (5b)^3$
$= 8a^3 - 60a^2b + 150ab^2 - 125b^3.$

EXERCISE XXXVII

Expand the following expressions:

3. $(x+y)^3$.	**7.** $(c+1)^3$.	**11.** $(3x^2-2z)^3$.
4. $(m-t)^3$.	**8.** $(a-3)^3$.	**12.** $(1+2m)^3$.
5. $(2x-y)^3$.	**9.** $(d^2+c^2)^3$.	**13.** $(3a^2-2b^5)^3$.
6. $(z-3y)^3$.	**10.** $(2yz-5)^3$.	**14.** $(-5-v)^3$.

15. What is the difference in meaning between "the cube of the sum of two numbers" and "the sum of the cubes of two numbers"? Illustrate, using 3 and 4 as the two numbers.

16. Give a rule for finding the cube of the difference of two numbers.

17. Expand $(c+d)^3$, also $(-c-d)^3$. If we change the signs of an expression, do we change the signs in its cube? Why?

II. FACTORING

58. Definitions. In a broad sense, any two or more numbers whose product is a given number are factors of that number. Thus, since $\frac{1}{3}\cdot\frac{6}{5}\cdot 15=6$, therefore $\frac{1}{3}$, $\frac{6}{5}$, and 15 are factors of 6; so also are $\frac{5}{12}$, 18, and $\frac{4}{5}$; the *important* factors of 6 are, however, 2 and 3.

In order to exclude fractional and other unimportant factors, we shall (as it is customary to do) define factors thus:

The **factors** of a number or algebraic expression are its rational * integral exact divisors.

E.g., the factors of $3x(a^2-b^2)$ are 3, x, $a+b$, and $a-b$, as well as the product of any two or more of these.

Observe, too that if 3, $a+b$, etc., are factors of any given expression, then -3, $-(a+b)$, etc., also are factors of this expression.

A factor (or expression) is said to be **prime** if it contains no factors except itself and 1; otherwise, it is **composite**.

* An expression is *rational* with regard to a particular letter if it contains no indicated root of that letter (see § 113).

By **factoring** a number (or expression) is usually meant the process of separating it into its prime factors.

Factoring an expression, as will appear later, often greatly simplifies algebraic work; it is therefore important that the pupil should early master those cases of factoring which present themselves most frequently.

59. Factors of a monomial. The literal factors of a monomial are evident by inspection, and the factors of the numerical coefficient are found as in arithmetic.

E.g., the factors of $30\,ax^3y^2$ are $2, 3, 5, a, x^3$, and y^2. (The x-and y-factors are as evident in the forms x^3 and y^2 as from $x \cdot x \cdot x$ and $y \cdot y$.)

60. Monomial and polynomial factors of a polynomial. If a polynomial contains a monomial factor, the latter is readily discovered by mere inspection.

E.g., in $12a^2x^3+4abx^2y-8ax^2y^2$, it is seen that each term contains the factor $4\,ax^2$, hence (see § 38) the other factor is $3\,ax+by-2\,y^2$; *i.e.*, $12\,a^2x^3+4\,abx^2y-8\,ax^2y^2=4\,ax^2\,(3\,ax+by-2\,y^2)$.

To factor a polynomial *completely* requires (1) the removal of all monomial factors, and (2) the factoring of the polynomial thus freed from its monomial factors. The simpler cases of (2) are considered in the next few articles; (1) may always be accomplished as above.

EXERCISE XXXVIII

Factor:

1. $6\,a^2x^3$.
2. $15\,mp^4z^3$.
3. $42\,s^4t^3$.
4. $210\,y^2v^5$.
5. $408\,m^3x^4y^2$.
6. $572\,a^2c^3wv^2$.

7. The expression $5\,a-10\,b+30\,x^2$ has what monomial factor? what polynomial factor? How do you find the former? the latter?

Separate the following expressions into their monomial and polynomial factors, and check your results:

8. $17\,x^2 - 51\,x^3$.

9. $4\,x^3 - 6\,x^2y$.

10. $4\,a^3b^2 - 26\,a^2b^3$.

11. $10\,m^4n^2 - 15\,m^3n^3$.

12. $-16\,x^2 - 2\,abx$.

13. $15\,x^4 - 10\,x^3 + 25\,x^2$.

14. $3\,a^5 - 6\,a^4b + a^4b^2$.

15. $m^7n^2 + m^5n^5 + m^2n^6$.

16. $3\,r^5 - 12\,r^3s^2 + 6\,rs^4$.

17. $ac - bc - cd - abcd$.

18. $32\,x^3y^3 - 28\,x^2y^2 + 12\,xy$.

19. $14\,x^2y^3z^3 - 2\,x^3y^2z^2 + 8\,xy^2z^2$.

20. $60\,m^2n^3r^2 - 45\,m^3n^2r^3 + 90\,m^4n^3r^2$.

21. $12\,x^2b^2y - 18\,xy^3b + 24\,x^4y^4b^4$.

22. $14\,a^2mn^2 - 21\,a^3m^2n^3 - 49\,a^4mn^2$.

23. $35\,c^2dx^3 + 5\,c^3d^2x^4 - 55\,c^2d^2x^5$.

24. $51\,xy^2z^3 - 68\,x^3y^2z^2 + 85\,x^4y^3z^4$.

25. $52\,a^2b^3c^4 - 65\,a^3b^2c^2 + 91\,a^2b^2c^2$.

26. Write $(m+n)^3 - 3(m+n)^4 + (m+n)$ as the product of two factors, one of which is $m+n$.

27. Write $2(3\,x-1)^2 - 5(3\,x-1) + 4(3\,x-1)^3$ as the product of two factors; also $6\,(2-a)^5 - 8\,(2-a)^3 - 12\,(2-a)^6$; also $x^2(a-c) - (1-3\,x^2)\,(a-c) - (a-c)$.

28. If $-5\,m^3n^2$ is one factor of $10\,m^4n^2 - 15\,m^3n^3$, what is the other (cf. Ex. 11)? Factor again the expressions in Exs. 12–16, in each case taking the monomial factor as *negative*.

61. Factoring by means of type forms. Expressions of the type $a^2 + 2\,ab + b^2$. Factoring being the inverse of multiplication, it follows that to every case of multiplication there corresponds a case of factoring. Ease in factoring, as in every inverse process, depends upon a ready knowledge of the corresponding direct process.

Thus, if we promptly recognize the form

$$a^2 + 2\,ab + b^2, \qquad \text{[see § 52}$$

then we can as promptly write down its factors, viz.:

$$a + b \text{ and } a + b.$$

So, too, the factors of

$$a^2 - 2\,ab + b^2$$

are

$$a - b \text{ and } a - b. \qquad \text{[see § 52}$$

The expressions $6\,mn + m^2 + 9\,n^2$ and $4\,x^2 + 25 - 20\,x$ belong to this type form, for, in each case, two terms of the trinomial are the squares of certain numbers, and the third term is twice the product of these numbers. These expressions may, therefore, be written as $(m + 3\,n)(m + 3\,n)$ and $(2\,x - 5)(2\,x - 5)$, or as $(m + 3\,n)^2$ and $(2\,x - 5)^2$, respectively.

EXERCISE XXXIX

Factor the following expressions:

1. $x^2 - 2\,bx + b^2$.
2. $u^2 + 2\,uw + w^2$.
3. $x^2 - 6x + 9$.
4. $y^2 - 4\,y + 4$.
5. $1 + 2\,a + a^2$.
6. $m^2 - 10\,m + 25$.
7. $49 - 14\,s + s^2$.
8. $a^2b^2 - 2\,ab + 1$.
9. $1 - 12\,y + 36\,y^2$.
10. $x^6 - 4\,x^3 + 4$.
11. $30\,x^5 + 225 + x^{10}$.
12. $9\,x^2 - 12\,xyz + 4\,y^2z^2$.
13. $6\,abcd + 9\,c^2d^2 + a^2b^2$.
14. $4 - 36\,a^2b^2 + 81\,a^4b^4$.

15. What first suggests to you that $x^2 + 9\,y^2 + 6\,xy$ may be the square of a binomial? How do you test the correctness of this supposition? When is a trinomial the square of a binomial?

16. Write out a carefully worded rule for factoring expressions of the types $a^2 + 2\,ab + b^2$ and $a^2 - 2\,ab + b^2$? How do we find the *terms* of the binomial? How do we determine the sign by which they are to be connected?

17. Is $a^4 + 2\,a^2b^3 - b^6$ the square of a binomial? Explain.

Factor the following expressions, and check your work.

18. $5\,ax^2 - 80\,ax + 320\,a$. [Remove monomial factor first.]
19. $7\,n^3 + 14\,abn^2 + 7\,a^2b^2n$.
20. $18\,a^3y - 60\,a^2by + 50\,ab^2y$.
21. $27\,c^4n^3 - 36\,c^3dn^3 + 12\,c^2d^2n^3$.
22. $-x^2 + 2\,xy - y^2$.
23. $-m^4 + 2\,m^3n^2 - m^2n^4$.
24. $x^{2n} + 4\,x^ny^3 + 4\,y^6$.
25. $a^{4p} - 6\,a^{2p}b^n + 9\,b^{2n}$.
26. $\frac{1}{4}\,c^2 - \frac{1}{3}\,c + \frac{1}{9}$.
27. $1 + \frac{9}{16}\,s^2 + \frac{3}{2}\,s$.
28. $m^{2p+2} + 2\,m^{p+1}n^{q+3} + n^{2q+6}$.
29. $(a + x)^2 + 2(a + x) + 1$.
30. $16 - 8(x + y) + (x + y)^2$.
31. $9(m - n)^2 - 6\,x(m - n) + x^2$.

62. Expressions of the type $a^2 - b^2$. From § 53 it follows that the factors of $a^2 - b^2$ are $a + b$ and $a - b$.

Again, the expression $25\,n^2 - 9\,t^2$ is of the above type, and its factors are $5\,n + 3\,t$ and $5\,n - 3\,t$.

EXERCISE XL

Factor the following expressions:

1. $y^2 - z^2$.
2. $y^2 - 9\,z^2$.
3. $4\,y^2 - 49\,b^2$.
4. $25\,a^2b^2 - 16$.
5. $9\,y^2 - 1$.
6. $225\,x^4 - 9\,y^6$.
7. $a^2x - b^2x$.
8. $36\,a^2e^2 - 81\,d^2$.
9. $x^{2n} - 4$.
10. $121\,a^4 - 36\,b^4$.
11. $64\,x^2y^{2n} - 144\,z^2$.
12. $(x + y)^2 - \frac{1}{4}$.
13. $49 - 36\,x^2y^2$.
14. $m^{2n} - n^{2m}$.
15. $49\,x^2y^5 - 16\,y^3d^8$.
16. $64\,x^2y^4 - 81$.
17. $289\,x^6z^9 - y^{8n}z$.
18. $4\,d^2 - (x - y)^2$.

19. In factoring the difference of two squares (*e.g.*, $a^2 - b^2$), how are the *terms* of each factor found? How are these terms connected in the first factor? in the second?

20. Write a rule for factoring the difference of two squares.

By rearranging and grouping terms factor the following expressions, and check your work:

21. $b^2 - 2\,bc - d^2 + c^2$.
[*i.e.*, $b^2 - 2bc + c^2 - d^2$, *i.e.*, $(b - c)^2 - d^2$].
22. $c^2 + 2\,cd + d^2 - e^2$.
23. $x^2 - b^2 - 2\,xy + y^2$.
24. $x^2 + 4\,xy - 4\,z^2 + 4\,y^2$.
25. $m^2 - 6\,m + 9 - p^2$.
26. $1 - s^2 - 2\,st - t^2$.
27. $25 - m^2 + 2\,mn - n^2$.
28. $-a^4 + b^4 - 2\,a^2 - 1$.
29. $-18\,k + 81 + k^2 - 25\,h^2$.
30. $-9\,u^2 + 49 - 12\,uv - 4\,v^2$.
31. $8\,c^2d^2 - 4 + c^4 + 16\,d^4$.
32. $s^2 - 4\,r^2 + t^2 - 2\,st$.
33. $-s^2 + t^2 - 4\,r^2 - 4\,rs$.
34. $36\,c^2 - a^2x^2 - 36 + 12\,ax$.
35. $-22\,xy + 121 - z^2 + x^2y^2$.

36. Supply the required factor in each of the following: $a^2 - b^2 = (-a + b)\cdot(\ ?\ )$; $16\,x^2 - 9\,y^2 = (-4\,x - 3\,y)\cdot(\ ?\ )$.

May the factors in Ex. 4 be written $(-5\,ab - 4)(-5\,ab + 4)$? Explain (cf. §§ 18, 58).

63. Expressions of the type $x^2 + mx + n$. From § 54 it follows that the factors of $x^2 + (a+b)x + ab$ are

$$x + a \text{ and } x + b.$$

Again, since the expression $k^2 + 7k + 12$ may be written in the form $k^2 + (3+4)k + 3 \cdot 4$, therefore its factors are

$$k + 3 \text{ and } k + 4.$$

So, too, $\quad p^2 + 2p - 15 = p^2 + (5-3)p + 5 \cdot (-3)$

$$= (p+5)(p-3).$$

From these illustrations we see that we can separate the trinomial $x^2 + mx + n$ into two binomial factors whenever we can separate n into two factors whose sum is m. Hence, whenever such an expression as $x^2 + mx + n$ can be factored, its factors may be found by a few trials — the number of trials never exceeding the number of pairs of factors of n.

EXERCISE XLI

1. If the expression $x^2 + 5x - 36$ is the product of two binomial factors, what is the product of the unlike terms in these two binomials? Have these terms like or unlike signs? Why? What is the sum of these unlike terms? Is the larger of them positive or negative? Why?

Factor the following expressions and check the work:

2. $x^2 - 3x + 2$.
3. $x^2 + x - 6$.
4. $x^2 - x - 2$.
5. $s^2 + 12s + 36$.
6. $y^2 + 6y + 5$.
7. $a^2 + 7a - 30$.
8. $p^2 - 12p + 35$.
9. $m^2 + 15m + 50$.
10. $k^2 - 3k - 40$.
11. $v^2 - 7v - 18$.
12. $t^2 + 13t - 30$.
13. $x^2 - ax - 90a^2$.
14. $v^2 - 11uv + 18u^2$.
15. $x^3 - 17x^2 + 72x$ [*i.e.*, $x(x^2 - 17x + 72)$].
16. $2x^2 - 6x + 4$.
17. $v^4 - 14v^3 - 32v^2$.
18. $ax^2 + 7a^2x + 6a^3$.
19. $66 + 39y^3 + 3y^6$.
20. $-a^2 - 27 + 12a$.
21. $a^2b^2 - 7ab + 10$ [*i.e.*, $(ab)^2 - 7(ab) + 10$].

22. $4x^2 + 4x - 3$ [*i.e.*, $(2x)^2 + 2(2x) - 3$].

23. $4x^2 - 8x - 21$.

24. $9x^2 + 6x - 8$.

25. $9x^2 - 21x - 8$.

26. $9x^2 + 27x + 14$.

27. $16x^2 - 56x + 33$.

28. $15 + 32x + 16x^2$.

29. $25x^2 - 8 - 10x$.

30. $t^{2n} - 24t^n + 63$.

31. $m^2 - 11mn + 28n^2$.

32. $s^2 - st - 42t^2$.

33. $-12xyz + x^2y^2 + 32z^2$.

34. $(m+n)^2 + 7(m+n) + 6$.

35. $(s-k)^2 - 26(s-k) + 69$.

36. $6 - y - y^2$.

37. $r^2 - (b-f)r - bf$.

38. $x^2 + (3a - 2b)x - 6ab$.

64. Expressions of the type $kx^2 + mx + n$. Every trinomial of this type which is the product of two binomials, may be readily factored by an extension of the method of § 63.

For example, to factor $6x^2 - 11x - 35$, we proceed thus:

$$\begin{aligned} 6x^2 - 11x - 35 &= \tfrac{1}{6}(36x^2 - 66x - 210) \\ &= \tfrac{1}{6}[(6x)^2 - 11(6x) - 210] \\ &= \tfrac{1}{6}(6x - 21)(6x + 10) \qquad [\S\ 63 \\ &= (2x - 7)(3x + 5). \end{aligned}$$

The given expression is first multiplied by 6 so as to make the first term an exact square, and the factor $\frac{1}{6}$ is then inserted so as to keep the value unchanged.

NOTE TO THE TEACHER. The above method may, if the teacher prefers, be replaced by the following; in that case § 64 should follow § 67.

Let $6x^2 - 11x - 35 = (ax + b)(cx + d)$, wherein a, b, c and d are to be determined;

then $6x^2 - 11x - 35 = acx^2 + (ad + bc)x + bd$,

whence $-11 = ad + bc$ and $6(-35) = ac \cdot bd$ (*i.e.*, $ad \cdot bc$).

If, therefore, we separate $6(-35)$, *i.e.*, -210, into two *factors* whose *sum* is -11, we shall then have found ad and bc; these factors are -21 and 10, hence we have

$$\begin{aligned} 6x^2 - 11x - 35 &= 6x^2 + (-21 + 10)x - 35 \\ &= 6x^2 - 21x + 10x - 35 \\ &= 3x(2x - 7) + 5(2x - 7) \\ &= (2x - 7)(3x + 5). \end{aligned}$$

When this method is used with young pupils, special care will be needed to keep the work from becoming merely mechanical.

EXERCISE XLII

1. In factoring $3x^2+13x+14$ by the method of § 64, what multiplier should be used? Why? What divisor must then be used? Why?

Factor the following expressions and check your work:

2. $3x^2+13x+14$.

3. $6a^2-11a+4$.

4. $3z^2+z-10$.

5. $4x^2+16x+15$.

6. $10y^2-13y-3$.

7. $9x^2+7x-2$.

8. $10x^2+x-2$.

9. $12x^2+4x-5$.

[Multiply and divide by 3.]

10. $18s^2-9s-5$.

[Multiply and divide by 2.]

11. $12m^2+7m-10$.

12. $20m^2-7m-6$.

13. $2a^2+a-55$.

14. $8t^2+7t-18$.

15. $10y^2+7y-12$.

16. $8n^2+14n-15$.

17. $6p^2-29p+35$.

18. $10b^2+37b-12$.

19. $10-19x+6x^2$.

20. $56x+15+20x^2$.

21. $8c^2-10cd-3d^2$.

22. $-28+39s-8s^2$.

23. $-30t^2-19t+5$.

24. $12p^2-28p+11$.

25. $16x^5+4x^3y^2-30xy^4$.

26. $4ab^2-73abc+18ac^2$.

27. $-14y-16+15y^2$.

28. $15x^{2n}+16x^ny+4y^2$.

29. $14k^{4c}-27k^{2c}-20$.

30. $3(a+b)^2+10(a+b)-8$.

31. $5(c-d)^2-7(c-d)-6$.

32. $15x^{2p}-x^p-28$.

33. $sx^2+(7s-t)x-7t$.

34. $cz^2+(fc-d)z-fd$.

35. $lx^2-lkx-mx+km$.

36. $6ay^2+2aby-3cy-bc$.

37. $90xyz^2-98a^2xyz+8a^4xy$.

38. Is a product altered when two of its factors are changed in sign? Explain (cf. § 18, also Ex. 36, p. 81). Change the signs in each factor found for Ex. 2 above, and thus write the factors of $3x^2+13x+14$ in a new form. Similarly, in each of Exs. 3–8 write the factors in a new form.

65. Squares of polynomials. Cubes of binomials. These types may be recognized by comparing them with the formulas of § 56 and § 57.

Thus, since the expression $x^2 + z^2 - 4yz + 2xz + 4y^2 - 4xy$ consists of three square terms and three double products, it *may* be the square of a trinomial. On rearranging its terms thus: $x^2 + 4y^2 + z^2 - 4xy + 2xz - 4yz$, and comparing with § 56, we see that the given expression is the square of $x - 2y + z$.

Again, the expression $12am^2 - 6a^2m - 8m^3 + a^3$, consisting, as it does, of four terms, two of which are cubes, *may* be the cube of a binomial; further examination shows that it *is* the cube of $a - 2m$.

EXERCISE XLIII

1. Is $a^2 - 2ab + c^2 + 2bc - 2ac + b^2$ the square of a trinomial? What suggests to you that it *may* be? How do you find the terms of this trinomial? Which of them are alike in sign? Which unlike in sign? Why?

Factor, and check your results as the teacher directs:

2. $m^2 + n^2 + s^2 + 2mn - 2ms - 2ns$.

3. $4x^2 + y^2 + 2yz + 4xy + z^2 + 4xz$.

4. $9v^2 + 2kx + x^2 - 6kv - 6vx + k^2$.

5. $6ac + 8bc + 9a^2 + c^2 + 24ab + 16b^2$.

6. $4c^2 + 9a^2 - 12ac + 16bc - 24ab + 16b^2$.

7. $1 + 2r - 2m + m^2 - 2rm + r^2$.

8. $2lm - 2ln + p^2 - 2lp + m^2 + l^2 - 2mn - 2mp + n^2 + 2np$.

9. If an expression (*e.g.*, $3pq^2 - q^3 + p^3 - 3p^2q$) is the cube of a binomial, how do you find the terms of this binomial? By what sign do you connect them? Illustrate.

Factor, and check by § 25:

10. $a^3 - 3a^2y + 3ay^2 - y^3$.

11. $m^3 - t^3 + 3mt^2 - 3m^2t$.

12. $3a^4 + 1 + 3a^2 + a^6$.

13. $-y^3 - 12x^2y + 6xy^2 + 8x^3$.

14. $-27y^2z + 27y^3 - z^3 + 9yz^2$.

15. $8 - c^6 - 12c^2 + 6c^4$.

16. $x^6 - 2x^5 + 10x^2 + x^4 - 10x^3 + 25$.

17. $216 - 108s^3t^2 + 18s^6t^4 - s^9t^6$.

18. $25 + 6m^2n - 10n + 9m^4 - 30m^2 + n^2$.

66. Factoring the type forms $x^n - y^n$ and $x^n + y^n$. By actual division we obtain the following results:

$$\text{I.}\quad \begin{cases} (x^2 - a^2) \div (x - a) = x + a, \\ (x^3 - a^3) \div (x - a) = x^2 + ax + a^2, \\ (x^4 - a^4) \div (x - a) = x^3 + ax^2 + a^2x + a^3, \\ (x^5 - a^5) \div (x - a) = x^4 + ax^3 + a^2x^2 + a^3x + a^4, \text{ etc.} \end{cases}$$

$$\text{II.}\quad \begin{cases} (x^2 - a^2) \div (x + a) = x - a, \\ (x^3 - a^3) \div (x + a) \text{ not exactly divisible,} \\ (x^4 - a^4) \div (x + a) = x^3 - ax^2 + a^2x - a^3, \\ (x^5 - a^5) \div (x + a) \text{ not exactly divisible, etc.} \end{cases}$$

$$\text{III.}\quad \begin{cases} (x^2 + a^2) \div (x - a) \text{ not exactly divisible,} \\ (x^3 + a^3) \div (x - a) \text{ not exactly divisible,} \\ (x^4 + a^4) \div (x - a) \text{ not exactly divisible, etc.} \end{cases}$$

$$\text{IV.}\quad \begin{cases} (x^2 + a^2) \div (x + a) \text{ not exactly divisible,} \\ (x^3 + a^3) \div (x + a) = x^2 - ax + a^2, \\ (x^4 + a^4) \div (x + a) \text{ not exactly divisible,} \\ (x^5 + a^5) \div (x + a) = x^4 - ax^3 + a^2x^2 - a^3x + a^4, \text{ etc.} \end{cases}$$

These quotients illustrate the following principles (for proofs see Exs. 17–19, p. 94):

(i) From I, *$x^n - a^n$ is always exactly divisible by $x - a$; the quotient terms are all positive.*

(ii) From II, *$x^n - a^n$ is exactly divisible by $x + a$ only when n is even; the quotient terms are alternately positive and negative.*

(iii) From III, *$x^n + a^n$ is never exactly divisible by $x - a$.*

(iv) From IV, *$x^n + a^n$ is exactly divisible by $x + a$ only when n is odd; the quotient terms are alternately positive and negative.*

(v) *The order of the letters and exponents is the same in all the quotients; the exponent of the first letter decreasing, and that of the second increasing, in passing toward the right.*

EXERCISE XLIV

Write the following quotients by inspection and then verify them by actual division:

1. $\dfrac{x^2 - y^2}{x - y}$.
2. $\dfrac{x^3 - y^3}{x - y}$.
3. $\dfrac{a^4 - b^4}{a - b}$.
4. $\dfrac{m^6 - n^6}{m + n}$.
5. $\dfrac{u^8 - v^8}{u - v}$.
6. $\dfrac{u^8 - v^8}{u + v}$.
7. $\dfrac{x^3 + y^3}{x + y}$.
8. $\dfrac{x^5 + y^5}{x + y}$.
9. $\dfrac{m^7 + s^7}{m + s}$.
10. $\dfrac{a^9 + b^9}{a + b}$.
11. $\dfrac{(x^2)^5 + (y^2)^5}{x^2 + y^2}$.
12. $\dfrac{(a^3)^2 - (c^3)^2}{a^3 - c^3}$.
13. $\dfrac{x^{10} - y^{10}}{x^2 - y^2}$. $\left[i.e., \dfrac{(x^2)^5 - (y^2)^5}{x^2 - y^2}\right]$.
14. $\dfrac{s^{10} + t^{10}}{s^2 + t^2}$.
15. $\dfrac{s^{10} - y^{10}}{s^5 - y^5}$.
16. $\dfrac{x^{12} - y^{12}}{x^4 - y^4}$.
17. $\dfrac{x^{12} - y^{12}}{x^3 - y^3}$.

18. In Exs. 5–11, above, express the dividend as the product of the quotient and the divisor.

19. Of which of the following binomials is $r - s$ a factor: $r^8 + s^8$; $r^{10} - s^{10}$; $r^7 - s^7$; $r^{11} + s^{11}$? Answer the same question for the factor $r + s$.

Write each of the following as the product of two factors:

20. $m^3 - n^3$.
21. $d^5 + e^5$.
22. $x^4 - y^4$.
23. $k^7 - l^7$.
24. $y^3 + z^3$.
25. $a^{10} + b^{10}$. [Cf. Ex. 14.]
26. $x^6 + y^6$.
27. $r^9 - s^9$.
28. $y^3 + 8$ [*i.e.*, $y^3 + (2)^3$].
29. $x^3 + 27$.
30. $8\,m^3 - 1$.
31. $t^5 - 32$.
32. $16\,p^4 - q^4$.
33. $32\,x^5 + 1$.
34. $8 - 27\,r^3$.
35. $16\,x^4 - 81$.
36. $27\,v^3 - 64\,w^6$.
37. $y^5 + 32\,x^{10}$.
38. $64\,r - r^7$.

39. Factor $a^6 - c^6$ in two ways: (1) by taking out the factor $a - c$, (2) by using § 53 (cf. Ex. 12, above) and then refactoring the two factors thus found. Which is the better plan to use when the *prime* factors of $a^6 - c^6$ are sought? Show that this plan is advisable in general, *e.g.*, with $x^8 - y^8$ and $p^{20} - q^{20}$.

Resolve the following expressions into their prime factors:

40. $x^4 - y^4$.

41. $a^6 - b^6$.

42. $a^8 - b^8$.

43. $m^8 - 1$.

44. $m^{12} - n^{12}$.

45. $a^{10}x^{10} - y^{10}$.

46. $64a^6 - 1$.

47. $a^8 - 81$.

48. $81a^4b^4 - 16x^4y^4$.

49. $x^9 - y^9$.

50. $x^9 + y^9$.

51. $x^{13} - xy^{12}$.

52. $3as^{12} - 3at^{12}$.

53. $64x^6 + y^6$.

54. $p^9 + 1$.

67. Factoring by rearranging and grouping terms. A rearrangement and grouping of the terms of an expression will often reveal a factor which could not be easily seen before.

E.g.,
$$\begin{aligned} ax - 3by + bx - 3ay &= ax + bx - 3by - 3ay \\ &= x(a+b) - 3y(a+b) \\ &= (a+b)(x-3y), \end{aligned}$$

i.e., $ax - 3by + bx - 3ay = (a+b)(x-3y)$.

Again,
$$\begin{aligned} x(x+4) - y(y+4) &= x^2 + 4x - y^2 - 4y \\ &= x^2 - y^2 + 4(x-y) \\ &= (x-y)(x+y) + 4(x-y) \\ &= (x-y)(x+y+4), \end{aligned}$$

i.e., $x(x+4) - y(y+4) = (x-y)(x+y+4)$.

EXERCISE XLV

Factor the following expressions and check your work:

1. $cx - cy + 3x - 3y$.

2. $ay + kx + ax + ky$.

3. $p^3 - p^2 + 7p - 7$.

4. $p^3 - p^2 - 7p + 7$.

5. $ac + bd - ad - bc$.

6. $9cy - 6cx - 12mx + 18my$.

7. $a^2c^2 - acd - abc + bd$.

8. $7mr - 3rs + 21ms - 9s^2$.

9. $5x^3 - x^2 + 2 - 10x$.

10. $5x^3 + 1 - x^2 - 5x$.

11. $ax^3 + 1 + a + x^3$.

12. $m^2 - n^2 - (m-n)^2$.

13. $3xy(x+y) + 16(x^3 + y^3)$.

14. $x^4 - xy^3 - ax^2 + ay^2$.

15. $ab + bx^n - x^ny^m - ay^m$.

16. $a^2 - 9x^2 + 4c^2 - 4ac$
[*i.e.*, $(a^2 - 4ac + 4c^2) - 9x^2$].

17. $-44k^2 - 49b^2 + 4k^4 + 121$.

18. $ac^2 + bd^2 - ad^2 - bc^2$.

19. $1 + ds - (c^2 + cd)s^2$.

20. $(a+1)^2 - (4a+3)^2$.

21. $(p^2 - q^2)^2 - (p^2 - pq)^2$.

22. $a^2x + abx + ac + b^2y + aby + bc.$

23. $(x^2 + 6x + 9)^2 - (x^2 + 5x + 6)^2.$

24. $x^2 - a^2 + y^2 - b^2 + 2xy - 2ab.$

25. $h^2 - m^2 + 10m + k^2 - 25 - 2hk.$

26. $(x + y)^2 + 12(x + y) - 85$ (cf. Exs. 34–35, p. 83).

27. $x^2 + 4xy + 4y^2 + 3x + 6y + 2$ (cf. Ex. 26 above).

28. $4x^2 + 10x - 6 - 5a - 4ax + a^2.$

29. Show that by changing the signs of two of them at a time the factors in Ex. 10 may be written in three different forms (cf. Ex. 38, p. 84). Is the same true in Ex. 18?

68. Factoring by means of other devices. It often happens that the factors of an expression will become apparent by adding a certain number to, and subtracting the same number from, the given expression; this, of course, leaves the value of the expression unchanged.

Ex. 1. Find the factors of $x^4 + x^2 + 1$.

Solution. If the second term in this expression were $2x^2$ instead of x^2, then (§ 61) the expression could be written $(x^2+1)^2$; this suggests that x^2 be both added and subtracted, which gives

$$\begin{aligned} x^4 + x^2 + 1 &= x^4 + 2x^2 + 1 - x^2 \\ &= (x^2+1)^2 - x^2 \\ &= (x^2 + 1 + x)(x^2 + 1 - x), \qquad [\S\ 62 \end{aligned}$$

i.e.,

$$x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1).$$

EXERCISE XLVI

2. Find the factors of $a^4 + a^2b^2 + b^4$.

Suggestion. By the method of Ex. 1,

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2a^2b^2 + b^4 - a^2b^2 \\ &= (a^2+b^2)^2 - (ab)^2. \end{aligned}$$

3. Find the factors of $x^2 - 4x - 32$.

Suggestion. Here the first two terms, plus 4, form an exact square; this suggests the following arrangement:

$$\begin{aligned} x^2 - 4x - 32 &= x^2 - 4x + 4 - 32 - 4 \\ &= (x-2)^2 - 36. \end{aligned}$$

4. What must be added to x^4+3x^2+4 to make it an exact square? What must then be subtracted to leave the value unchanged? Factor the given expression.

5. Can the sum of two squares be factored (cf. § 66)? Is x^4+4 the sum of two squares? Can it be factored?

6. What must be added to x^4+4 to make it $(x^2+2)^2$? Is the added term a square? Factor x^4+4.

Factor:

7. p^4+q4^4.
8. $x^4+64\,y^4$.
9. $m^4+m^2n^2+n^4$.
10. $x^4+a^2x^2+a^4$.
11. $x^8+x^4y^4+y^8$.
12. $x^2+6\,x+5$.
13. $9\,s^2+30\,st+16\,t^2$.
14. $a^4b^4+a^2b^2c^2d^2+c^4d^4$.
15. $9\,x^4+8\,x^2y^2+4\,y^4$.
16. $4\,a^8-21\,a^4b^4+9\,b^8$.
17. $5\,x^4-70\,x^2y^2+5\,y^4$.
18. $9\,a^4+26\,a^2b^2+25\,b^4$.
19. $a^2+2\,ab-d^2-2\,bd$.
20. $4\,a^4+81$.
21. $x^5y^2+4\,xy^2$.
22. $m^5+4\,mn^4$.
23. $a^4+8\,a^2-128$.
24. $5\,nx^4-70\,nx^2+200\,n$.
25. Find the four factors of $x^4+y^4+z^4-2\,x^2y^2-2\,x^2z^2-2\,y^2z^2$.

69. General plan for factoring a polynomial.

1. By inspection, find and remove all monomial factors.

2. By comparison with type forms, by rearrangement and grouping of terms, or by some other device, separate the resulting polynomial factor into two factors.

3. Then, if possible, separate each of these factors into two others, and so continue until all factors are prime.

NOTE. By the above plan the simpler expressions can usually be factored. For determining the binomial factors of longer polynomials, see § 71.

EXERCISE XLVII

Factor:

1. $4\,ax^2-4\,ay^2$.
2. $49\,k-k^3$.
3. l^3p-p^4.
4. $s^3+16\,s^2+15\,s$.
5. $x^4-8\,x^3+15\,x^2$.
6. $m^2x^5+m^2y^5$.

7. $tv - nw - nv + tw$.

8. $v^3 - 7v^2 - 2v + 14$.

9. $1 - 24s + 144s^2$.

10. $a^2b^2 - 4ab^2y + 4b^2y^2$.

11. $p^3 - 8r^3$.

12. $1 - y^4$.

13. $c^2 - 5c - 14$.

14. $k^4 - 17k^2 + 16$.

15. $3x^2 - 10xy + 3y^2$.

16. $49m^2s^2 + 42mns + 9n^2$.

17. $c^2 + x^2 - 2cx - 1$.

18. $b^8 + b^4y^2 + y^4$.

19. $2u^3 - 14u^2 + 70 - 10u$.

20. $4c^2 - 25a^2 + b^2 - 4bc$.

21. Give two methods for checking an exercise in factoring. Illustrate, using Ex. 15 above.

Factor and check as the teacher directs:

22. $n^5 - 1$.

23. $q^{4p} - r^{6k}$.

24. $216 + y^3$.

25. $a^8 + 4$.

26. $3(x-y)^2 - 27$.

27. $rsv^2 - ar^2st - 4cr^3$.

28. $x^2 + ax - ay - yx$.

29. $m(d^2 - 3) + d^2 - 3$.

30. $m^{2a} - 4m^ab + 4b^2$.

31. $k(l^2 - 4) - l^2 + 4$.

32. $m^4n^4 + 4 - 5m^2n^2$.

33. $7(x+a) - 11(x^2 - a^2)$.

34. $y^2 - y + \frac{1}{4}$.

35. $-xy - z(3y - x - 3z)$.

36. $x^6 - y^6 - 3x^4y^2 + 3x^2y^4$.

37. $6x^{10} + 12x^5 - 18$.

38. $a^2x^3 - x^5$.

39. $s^{12} - t^{12}$.

40. $7r^3 - .007$.

41. $a^{15} + 1$.

42. $(a-b)^3 - c^3$.

43. $k^{16} + 4p^4$.

44. $12 + s(t^2 - 4) - 3t^2$.

45. $x^{2n-2} + 2x^{n-1}by + b^2y^2$.

46. $m^3 - 1 - 3m(m-1)$.

47. $3a^2p - 28rq - 21rp + 4a^2q$.

48. $(5a+y)^2 - 7(5a+y) + 10$.

49. $8 - 12mn + 6m^2n^2 - m^3n^3$.

50. $m^2 + 6mn - 16x^2y^2 + 9n^2$.

51. $(x-y)^2 - 2y + 2x + 1$.

52. $m^8n^3 + 2m^6n^7r^2s^3 + m^4n^{11}r^4s^6$.

53. $(a^2+5a+4)^2 - (a^2-5a-6)^2$.

54. $x^2 - 2xy + 1 + y^2 + 2(x-y)$.

55. $(c-3)^3 - 1 - 3(c-3)^2 + 3(c-3)$.

56. $m^2 - 2mn + n^2 - s^2 + 2st - t^2$.

57. $(c^2 - 2cd + d^2)^2 - (3c^2 - cd - 2d^2)^2$.

58. $x^2 + 9y^2 + 25z^2 - 6xy - 10xz + 30yz$.

59. $2(a^2b^2 - a^2c^2 - b^2c^2) + a^4 + b^4 + c^4$.

60. $a^2t^2 + b^2r - b^2s + a^2r + b^2t^2 - a^2s$.

61. $a^2 - 2ab + b^2 - 2ac + 2bc + c^2 - 2ad + 2bd + 2cd + d^2$.

62. $x^2 - 9x + 14 = (x-7)\cdot(\ ?\ ) = (7-x)\cdot(\ ?\ )$.

63. $c^3 - r^3 = (c-r)\cdot(\ ?\ ) = (r-c)\cdot(\ ?\ )$.

64. $r^3 - 36r = r(r+6)\cdot(\ ?\ ) = -r(r+6)\cdot(\ ?\ )$.

65. Write the four factors of $x^4 - 10x^2 + 9$ in seven different ways (cf. Ex. 29, p. 89).

70.* **Remainder theorem.** In Ex. 53, p. 52, it was seen that if $x^2 + 3x + 1$ is divided by $x - a$ the remainder is $a^2 + 3a + 1$; *i.e., the remainder is what the dividend would become if a were substituted for x.* (Cf. also Ex. 52, p. 52.)

And this relation between dividend and remainder is not accidental; it is true for *all* such expressions. For, let

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + Hx + K$$

be any polynomial in x, let it be divided by $x - a$, and let Q and R, respectively, represent the quotient and remainder; then

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \cdots + Hx + K = Q(x-a) + R.$$

Moreover, since the second member of this equation, when multiplied out, must be exactly like the first, therefore this equation is true for *all* values that may be assigned to x; but if the value a be given to x, the equation becomes

$$Aa^n + Ba^{n-1} + Ca^{n-2} + \cdots + Ha + K = R,\dagger$$

hence, *in every such division, the remainder may be obtained by simply substituting a for x in the dividend.*

71.* **Application of the remainder theorem to factoring.** By means of the remainder theorem (§ 70), and without actually performing the division, write down the remainder resulting from dividing $x^3 - 3x^2 + 3x + 2$ by $x - a$. Also write the remainder when $x^3 - 3x^2 + 3x - 2$ is divided by $x - 2$. What is

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* Articles 70 and 71, with Exercise XLVIII, may, if the teacher prefers, be omitted till the subject is reviewed.

† Since, in that case, $Q(x-a)$ becomes $Q'(a-a)$, *i.e.*, zero; and R is the same as before substituting, since it does not contain x.

the *value* of this last remainder? Does this show that $x-2$ is a factor of x^3-3x^2+3x-2?

Binomial factors of many polynomials may be found in this way, for, from § 70, it follows that if

$$Aa^n+Ba^{n-1}+Ca^{n-2}+\cdots+Ha+K=0,$$

then, and then only, is $Ax^n+Bx^{n-1}+Cx^{n-2}+\cdots+Hx+K$ exactly divisible by $x-a$; for in that case, and in that case only, is the remainder zero.

Thus, we know that $x-3$ is a factor of x^3-2x^2-4x+3 because $3^3-2\cdot 3^2-4\cdot 3+3=0$; and $x+1$, *i.e.*, $x-(-1)$, is a factor of x^2+7x+6 because $(-1)^2+7(-1)+6=0$.

Again, if $x-a$ is a factor of x^3-x^2-2x+8, then *a is a factor of 8*; hence, in seeking such factors of x^3-x^2-2x+8 we need try only 1, -1, 2, -2, 4, -4, 8, and -8 in place of a.

When, by any process whatever, any factor of an expression has been discovered, this factor may be divided out; the remaining factors may then be more easily found.

EXERCISE XLVIII

1. If $x^4+6x^2-12x+5$ is divided by $x-a$, what is the remainder? Without performing the division, find the remainder when the divisor is $x-2$; also when it is $x-1$ and when it is $x+1$. Which of these divisors is a factor of the given expression?

2. If the expression x^3-3x^2-x+3 has a factor of the form $x-a$, what are the four possible values of a? Find *all* the binomial factors of x^3-3x^2-x+3.

By the above method, factor the following expressions:

3. x^3-7x+6.

4. $x^3-9x^2+23x-15$.

5. $x^3+14x^2+35x+22$.

6. $x^3-11x^2+31x-21$.

7. $k^3+4k^2-11k-30$.

8. $w^4-15w^2+10w+24$.

9. $a^3+7a^2+2a-40$.

10. $c^3-5c^2-29c+105$.

11. $x^4-x^3-7x^2+x+6$.

12. $y^5-10y^4+40y^3-80y^2+80y-32$.

13. If $x-k$ is a factor of any given expression, what does the value of that expression become when $x=k$? Why? If any given expression becomes zero when $x=k$, is $x-k$ a factor of the expression? Why?

14. By means of the remainder theorem show that $a-b$, $b-c$, and $c-a$ are factors of $a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)$.

15. Write the remainder when $(2x-3a)^2+(3x-a)^3$ is divided by $x-a$; also the remainder when $(x-y+z)^3-y^3+x^3$ is divided by $x-y$; and by $x+y$, that is, by $x-(-y)$.

16. What is the remainder when x^5-a^5 is divided by $x-a$? Why? Write the remainder when x^5+a^5 is divided by $x-a$; when x^5+a^5 is divided by $x+a$.

17. By means of the remainder theorem, show that x^n-a^n is exactly divisible by $x-a$; also that x^n+a^n is not exactly divisible by $x-a$ (cf. § 66).

18. By means of the remainder theorem, show that x^n-a^n is exactly divisible by $x+a$ only when n is an even positive integer.

19. By means of the remainder theorem, show that x^n+a^n is exactly divisible by $x+a$ only when n is an odd positive integer.

72. Solving equations by factoring. Factoring greatly simplifies the solution of certain kinds of equations. The following examples illustrate the procedure.

Ex. 1. Given $x^2-5x+6=0$; to find its roots, *i.e.*, to find those values of x for which this equation is satisfied (cf. § 45).

Solution. By § 63 the first member of this equation is the product of $x-3$ and $x-2$; hence the equation may be written thus:

$$(x-2)(x-3)=0.$$

Now this equation is satisfied if either

$$x-2=0 \text{ or } x-3=0, \qquad [\S\ 41$$

i.e., if either $\quad x=2 \text{ or } \quad x=3.$

On substitution these values are found to check; they are, therefore, the roots of the given equation.

Ex. 2. Given $x^2 = 3x + 4$; to find its roots.

Solution. On transposing, this equation becomes

$$x^2 - 3x - 4 = 0,$$

i.e., $(x-4)(x+1) = 0;$ [§ 63

hence either $x - 4 = 0$ or $x + 1 = 0$,

i.e., $x = 4$ or $x = -1$;

and these numbers check, therefore the roots are 4 and -1.

Ex. 3. Solve the equation $6x^2 - 11x = 35$.

Solution. On transposing and factoring (§ 64), this equation becomes

$$(3x+5)(2x-7) = 0;$$

hence $3x + 5 = 0$ or $2x - 7 = 0$;

therefore the roots are $-\frac{5}{3}$ and $\frac{7}{2}$.

Remark. Since the roots of the equation $(x-a)(x-b) = 0$ are a and b, therefore an equation which shall have any given numbers as roots may be immediately written down; thus the equation whose roots are 3 and 8 is

$$(x-3)(x-8) = 0, \quad i.e., \ x^2 - 11x + 24 = 0.$$

Similarly, the equation whose roots are 2, -1, and 5 is

$$(x-2)(x+1)(x-5) = 0, \quad i.e., \ x^3 - 6x^2 + 3x + 10 = 0.$$

EXERCISE XLIX

4. What is meant by a root of an equation (cf. § 45)? May an equation have more than one root?

5. What values of x satisfy the equation $(x-2)(x-3) = 0$? Can any values of x other than 2 or 3 satisfy this equation? Explain. How many roots, then, has this equation?

Solve the following equations by factoring, and check the roots:

6. $y^2 - 6y + 5 = 0$.

7. $x^2 - 4x - 21 = 0$.

8. $s^2 - 13s + 40 = 0$.

9. $x^2 - 2x = 15$.

10. $k^2 + 4k = 45$.

11. $x^2 - 4 = 0$.

12. $m^2 - 36 = 0$.

13. $y^2 - 7y = 0$.

14. $c^2 + 22c = -121$.

15. $v^2 - 3v - 50 = 38$.

16. $3y^2 + y - 10 = 0.$

17. $6x^2 - x = 1.$

18. $4v^2 - 27 = 12v.$

19. $8y^2 + 15 = -26y.$

20. $5x^2 - 7x = 0.$

21. $12z^2 = -4z.$

22. $x^2 - 3ax - 54a^2 = 0.$

23. $s^2 - (c+d)s + cd = 0.$

24. $8x^2 + 10x = 3.$

25. $36 = -x^4 + 13x^2.$

26. $x^3 + x^2 - x = 1.$

27. $2x^3 + 5x^2 = 2x + 5.$

28. What are the roots of $(x-1)(x-2)(x+2) = 0$? Explain. Determine by inspection the roots of $(x+1)(3x-2) = 0.$

29. Determine by inspection the roots of:

(1) $(5x-3)(x-1) = 0.$
(2) $(y - \frac{3}{4})(2y+9) = 0.$
(3) $m(3m+1)(4m-3) = 0.$
(4) $(x-a)(2x-11a)(4x+5a) = 0.$

30. Write an equation whose roots are 5 and 2. Also one whose roots are 3, 1, and 7.

31. Write the equations whose roots are: 1 and -5; $\frac{2}{3}$ and 6; a and b; 3, -1, and 5; a, $-a$, and $2a$; 1, 2, 3, and 4.

The following problems lead to equations whose roots may be found by factoring. Solve and check each problem.

32. Find a number such that if 3 and 5 are subtracted from it in turn, the product of the two remainders is 24. How many solutions has this problem? Explain.

33. The sum of two numbers is 12, and the square of the larger is 1 less than 10 times the smaller. Find the numbers (cf. Ex. 18, p. 6).

34. The difference between two numbers is 2, and the sum of their squares is 130. What are these numbers?

35. One side of a rectangle is 3 feet longer than the other. If the longer side be diminished by 1 foot and the shorter side increased by 1 foot, the area of the rectangle will then be 30 square feet. How long is this rectangle?

NOTE. The equation of this problem has two roots, one positive and one negative; but only the positive root will satisfy the problem itself, for it is implied that the dimensions of the rectangle are positive.

36. How may $128 be divided equally among a certain number of persons so that the number of dollars received by each person shall exceed the number of persons by 8?

37. The senior class of a certain school present the school with a picture whose cost is $12. If each senior contributes 3 times as many cents as there are members in the class, how large is the class? How much does each member pay?

38. A rectangular orchard contains 2800 trees, and the number of trees in a row is 10 less than twice the number of rows. How many trees are there in a row?

39. If the dimensions of a certain rectangular box which contains 120 cubic inches were increased by 2, 3, and 4 inches, respectively, the new box would be cubical in form. Find the dimensions of this box (cf. § 71).

CHAPTER VII

HIGHEST COMMON FACTOR—LOWEST COMMON MULTIPLE

I. HIGHEST COMMON FACTOR

73. Definitions. A factor of each of two or more algebraic expressions (or numbers) is called a **common factor** of these expressions. The **highest common factor** (H. C. F.) of two or more expressions is the product of all the prime factors (§ 58) that are common to these expressions; it is, therefore, the factor of *highest degree* common to the given expressions.

Thus, the H. C. F. of $9\,a^3b^2n^2$ and $6\,ab^3s^4$ is $3\,ab^2$, because when this factor is removed from the given expressions they have no *common* factor left.

So, too, $2\,x(a-1)^2$ is the H. C. F. of $6\,a^2x^2\,(a-1)^4$ and $8\,x\,(a-1)^2(s-t)^3$.

Two or more algebraic expressions which have no common factor except unity are said to be **prime to each other.**

74. Highest common factor of two or more monomials. The H. C. F. of two or more monomials may, obviously, always be found by inspection.

E.g., to find the H. C. F. of $12\,a^3b^2xy$, $6\,ab^3x^2$, and $9\,ab^2x^4$.

Inspection shows that 3, a, b^2, and x are the only factors common to the given monomials; hence the H. C. F. of these monomials is $3\,ab^2x$.

A rule for writing down the H .C. F. of several monomial expressions may be formulated thus: *To the H. C. F. of the numerical coefficients annex those letters that are common to the given monomials, and give to each of these letters the lowest exponent which it has in any of the monomials.*

75. H. C. F. of polynomials whose factors are known. By first writing any given polynomials in their factored forms their H. C. F. may be found by inspection.

For example, to find the H. C. F. of $4\,ax^2-20\,ax+24\,a$ and $6\,abx^2+24\,abx-126\,ab$, we write:

$$4\,ax^2-20\,ax+24\,a=4\,a\,(x-2)\,(x-3), \qquad [\S\ 63$$

and $$6\,abx^2+24\,abx-126\,ab=6\,ab\,(x+7)\,(x-3);$$

hence their H. C. F. is $2\,a\,(x-3)$.

EXERCISE L

Find the H. C. F. of each of the following sets of expressions:

1. $3\,a^2b^3$ and $6\,ab^4$.
2. $15\,x^3y^2$, $24\,x^2y^4$, and $18\,x^4y$.
3. $16\,x^2y^3z^3$, $52\,y^4z^6$, and $39\,x^7y^8$.
4. $195\,a^4b^6c^7$ and $260\,a^5bc^4$.
5. $96\,y^7z^3$, $100\,y^2z^5$, and $56\,y^6z^4$.
6. $104\,x^my^{2n}z^{3r}$ and $364\,x^{2m}y^{3n}z^{4r}$.
7. $(c+d)^3(c-d)$ and $(c+d)(c-d)^2$.
8. $6\,(c+d)^2(c-d)^2$ and $15\,(c-d)^2(c+d)$.
9. $24\,a^3x\,(y-z)^2$ and $56\,a^2bx^3\,(y-z)^4$.
10. a^2-b^2, $a(a-b)$, and $a^2-2\,ab+b^2$.
11. $x^2+7\,x+10$ and $x^2+12\,x+20$.
12. m^2-m-12 and $m^2-4\,m-21$.
13. $15\,(yz-z)$ and $35\,(y^4z-yz)$.
14. Is $-(a-b)$, *i.e.*, $b-a$, a common factor of the expressions in Ex. 10 (cf. Ex. 36, p. 81)? May we then call the H. C. F. of these expressions either $a-b$ or $b-a$?
15. Show that the H. C. F. of m^2-mn and n^2-mn is either $m-n$ or $n-m$.

In each of Exs. 16–25 find two forms of the H. C. F.:

16. r^2-s^2 and s^3-r^3.
17. $5\,a-as$ and $3\,s^2-75$.
18. p^3-125 and $p^3-10\,p^2+25\,p$.

19. $x^3 + a^3$ and $3a^3 + 3a^2x - 5ax^2 - 5x^3$.

20. $28n^2 - 17n - 3$ and $4n^2 + 5n - 6$.

21. $5 - 19k - 4k^2$ and $k^2 + 2k - 15$.

22. $a^2x - x - y + a^2y$ and $a^4x + 4a^2x - 5x$.

23. $12ab^2x + 4ab^2x^2 - 40ab^2$, $18a^2mx^2 - 54a^2mx + 36a^2m$, and $6a^2x^2y - 6a^2xy - 12a^2y$.

24. $uv - u^2$, $u^2 - 5v + 5u - uv$, and $3u^2 - 10uv + 7v^2$.

25. $15a^4x^2 + 15a^2b^4x^2 + 15b^8x^2$ and $3(a^2 - ab^2 + b^4)$.

Find the H. C. F. of each of the following sets of expressions:

26. $2x^2 - x - 3$ and $2x^3 + 11x^2 - x - 30$.

SUGGESTION. Find the factors of $2x^2 - x - 3$ and determine by trial which of these are factors of $2x^3 + 11x^2 - x - 30$ also. This plan may be used whenever any *one* of a given set of expressions is easily factored.

27. $(x + 3)(x^2 - 4)$ and $x^4 + 4x^3 + 2x^2 - x + 6$.

28. $a^3 + 1$, $3a^3 - 4a^2 + 4a - 1$, and $2a^3 + a^2 - a + 3$.

29. $b^3 - 8$, $b^3 + b^2 + 2b - 4$, and $b^4 + 2b^3 - b^2 - 10b - 20$.

30. Of what is the H. C. F. of two or more expressions composed? State a rule for finding the H. C. F. of two or more expressions which may easily be separated into their prime factors.

31. Is the H. C. F. as above defined the same as the greatest common divisor (G. C. D.) in the arithmetical sense? What is the H. C. F. of $x^2(x - 1)^2$ and $x(x^2 - 1)$? Is this also the G. C. D. of these expressions for all values of x? Try $x = 3$, also $x = 4$.

76.* H. C. F. of polynomials neither of which is easily factored. The H.C.F. of two or more polynomials can always be found by what is known as the Euclidean (division) process. This process is essentially the same as that used in arithmetic to find the G.C.D. of two numbers.

The steps in the arithmetical process are: (1) Divide the larger number by the smaller; (2) if there is a remainder, divide the smaller number [*i.e.*, the divisor in step (1)] by this remainder;

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* Articles 76, 77, and 78, with Exercises LI and LII, may, if the teacher prefers, be omitted till the subject is reviewed.

(3) divide the remainder in (1) by the remainder in (2); (4) so continue, dividing each remainder by the one following, until there is no remainder; (5) the last divisor is the G.C.D. sought.

Thus, to find the G. C. D. of 1183 and 2639.

$$
\begin{array}{l}
1183)2639(2 \\
\quad\ \ \underline{2366} \\
\quad\ \ \ 273)1183(4 \\
\qquad\quad\ \ \underline{1092} \\
\qquad\qquad 91)273(3 \\
\qquad\qquad\quad\ \underline{273} \\
\qquad\qquad\qquad 0
\end{array}
$$

This work may be more compactly arranged thus:

	QUOTIENTS	
1183	2	2639
		2366
1092	4	273
91	3	273
		0

The last divisor, 91, is the G. C. D. of the given numbers.

Similarly, the H.C.F. of $x^4 + 3x^3 + 2x^2 - x - 5$ and $x^3 + x^2 - 2$ may be found thus:

	QUOTIENTS	
$x^4 + 3x^3 + 2x^2 - x - 5$	$x + 2$	$x^3 + x^2 - 2$
$x^4 + x^3 \quad - 2x$		$x^3 - x^2$
$2x^3 + 2x^2 + x - 5$		$2x^2 - 2$
$2x^3 + 2x^2 \quad - 4$		$2x^2 - 2x$
$x - 1$	$x^2 + 2x + 2$	$2x - 2$
		$2x - 2$
		0

Hence $x - 1$, the last divisor, is the H. C. F. of the given polynomials.

EXERCISE LI

By the above method, find the H. C. F. of each of the following pairs of expressions:

1. $x^2 + 5x + 6$ and $4x^3 + 21x^2 + 30x + 8$.
2. $6a^2 - 13a - 5$ and $18a^3 - 51a^2 + 13a + 5$.
3. $5m^2 - 2m - 3$ and $15m^3 - 6m^2 - 4m + 3$.
4. $c^3 - 2c^2 - 2c - 3$ and $c^4 - c^3 - 3c^2 - 4c - 2$.
5. $12x^4 - 8x^3 - 55x^2 - 2x + 5$ and $6x^3 - x^2 - 29x - 15$.
6. $18x^4 + 75x^3 + 17x^2 - 23x - 18$ and $6x^3 + 23x^2 - 3x - 10$.
7. $80y^5 + 16y^4 + 16y^3 - 8y^2 - 3y - 2$ and $20y^3 + 4y^2 - y - 3$.
8. $4k^4 + 20k^3 - 10k^2 - 43k + 35$ and $2k^3 + 11k^2 - 25$.
9. $5n^4 - 10n^3 + 11n^2 - 6n + 1$ and
   $10n^5 - 5n^4 - 7n^3 + 19n^2 - 14n + 2$.

77.* Proof of principle involved in § 76 (see footnote, p. 100). The success of the method employed in § 76 is due to the following considerations:

Let A and B represent any two polynomials in x, the degree of A being at least as high as that of B, and let q and R represent the quotient and remainder respectively, when A is divided by B; then

$$A = qB + R. \qquad \text{[Ex. 20, p. 50}$$

This equation shows that: (1) every divisor common to B and R is a divisor of A also (why?), and (2) every divisor common to A and B is a divisor of R also (why?); hence the H. C. F. of B and R is the same as that of A and B.

If now B is divided by R, giving p and M as quotient and remainder respectively, then, by reasoning as above, we see that the H. C. F. of M and R is the same as that of B and R, and therefore the same as that of A and B.

Suppose now that this series of divisions is continued; then, by the above reasoning, the H. C. F of A and B is the same as that of *any two successive remainders.*

If now the last one of this series of divisions is *exact, i.e.*, if the final remainder is zero, then the H. C. F. of the two preceding remainders is the last divisor itself; hence the *last divisor* is the H. C. F. of A and B, which was to be found.

Remark. The H. C. F. of two expressions is evidently not altered by multiplying (or dividing) either of them by any number which is not a factor of the other; this fact enables us to avoid fractional coefficients in the division process.

Thus, to find the H. C. F. of $3x^3 + 8x^2 + 3x - 2$ and $x^3 - 2x^2 + x + 4$:

$3x^3+8x^2+3x-2$	3	x^3-2x^2+x+4
$3x^3-6x^2+3x+12$		$x^3 \qquad -x$
$14)14x^2-14$		$-2x^2+2x+4$
x^2-1	$x-2$	$-2x^2 \qquad +2$
x^2+x		$2)2x+2$
$-x-1$	$x-1$	$x+1$
$-x-1$		
0		

Before beginning the second division the factor 14 is suppressed (see Remark above), and later 2 is suppressed also; fractional coefficients are thus avoided.

Hence $x + 1$, the last divisor, is the H. C. F. of the given expressions.

As a further illustration, let us find the H. C. F. of

$$x^4+4x^3+2x^2-x+6 \text{ and } 2x^3+9x^2+7x-6.$$

$$\begin{array}{r|c|r}
x^4+4x^3+2x^2-x+6 & x,\ +1 & 2x^3+9x^2+7x-6 \\
2 & & \underline{2x^3+10x^2+12x} \\
\hline
2x^4+8x^3+4x^2-2x+12 & & -x^2-5x-6 \\
\underline{2x^4+9x^3+7x^2-6x} & & \underline{-x^2-5x-6} \\
-x^3-3x^2+4x+12 & & 0 \\
\underline{-2} & & \\
2x^3+6x^2-8x-24 & & \\
\underline{2x^3+9x^2+7x-6} & & \\
-3)\,\underline{-3x^2-15x-18} & & \\
x^2+5x+6 & 2x-1 &
\end{array}$$

Before beginning the division the factor 2 is introduced so as to avoid fractional coefficients in the quotient; later -2 is introduced for the same purpose; and finally -3 is rejected.

Hence x^2+5x+6, the last divisor, is the H. C. F. of the given expressions.

78.* **Supplementary to § 77** (see footnote, p. 100). (i) If the polynomials whose H.C.F. is sought contain monomial factors, these should be set aside before the division process is begun. Monomial factors that are *common* to the given polynomials must, of course, be reserved; all others may be rejected.

(ii) The H. C. F. of three or more polynomials is found by first finding the H. C. F. of any two of them, then the H.C.F. of that result and the third polynomial, and so on until all the polynomials have been used.

(iii) To find whether polynomials which involve more than one letter have a common factor containing any particular one of these letters, they need only be arranged according to powers of that letter, and divided as already described. By a repetition of this process *all* the common factors of such polynomials, and hence, their H. C. F., may be found.

[For fuller discussion of H. C. F. see *El. Alg.* pp. 116–121.]

EXERCISE LII

Find, by the Euclidean method, the H. C. F. of:

1. x^3-3x^2+3x-1 and $x^4-2x^3+2x^2-2x+1$.

2. $c^4+4c^3-12c^2+c+6$ and $c^4-c^3-2c^2+c+1$.

3. $5z^3+18z^2-33z+10$ and $2z^3+5z^2-22z+15$.

4. $2x^3+3x^2+2x-2$ and $x^4+2x^3+x^2-2x-2$.

5. $r^5-2r^3+2r^2-4$ and $r^7+2r^4-r^3-2$.

6. $1 - 4m^3 + 3m^4$ and $1 - 5m^3 + 4m^4 + m - m^2$.

7. $63 + k^3 - 9k - 7k^2$ and $40k + k^4 - 5k^3 + 111 - 23k^2$.

8. $x^5 - 4x^3 - 2x^2 - 8 + x^4$ and $2x^5 + 9x^4 - x^3 - 4x^2 + 14x - 16$.

9. $8x^3 - 22x^2 + 17x - 3$ and $6x^3 - 17x^2 + 14x - 3$.

10. $2x^2 - 3x - 35$ and $x^4 + 14x^2 - 9x^3 + 35x - 25$.

11. What is meant by the H. C. F. of two expressions A and B? If a is not a factor of A, how does the H. C. F. of A and $a \cdot B$ compare with the H. C. F. of A and B? Explain.

12. If a is a factor of A, but not of B, how does the H. C. F. of A and $a \cdot B$ compare with the H. C. F. of A and B? In introducing and suppressing factors during the process of division (§ 77), what precaution must be exercised, and why?

Find the H. C. F. of the following expressions:

13. $m^4 - 3m^2 + 1$ and $m^7 - 2m - 2 - m^6 - m^5 + 2m^2$.

14. $a^7 + 2a^5 - 5a^2 - 10$ and $a^5 + a^3 - a^2 - 2a - 2$.

15. $x^5 - 4x^4 - 2 + 3x - 3x^2 + 5x^3$ and $x + 2x^3 + 2 - 5x^2$.

16. $s^5 - 2s^4 - 2s^3 - 11s^2 - s - 15$ and $2s^5 - 7s^4 + 4s^3 - 15s^2 + s - 10$.

17. $x^4 + 3x^3 - 2x^2 - 6x$ and $4x^2 - x + x^5 + 4x^4 - 12 + 4x^3$.

18. $21ax - 17ax^2 - 5ax^3 + ax^4$ and $5ax^3 - 34ax^2 - 7ax$.

19. $7m^2x^3 - 49m^2x + 42m^2$ and $14a^2mx^3 + 14a^2mx^2 - 56a^2mx - 56a^2m$.

20. $48s^3tx^4 - 162s^3tx^2 + 54s^3t$ and $18s^2t^2u - 9st^2ux - 48s^2t^2ux^2 + 24s^2t^2ux^3$.

21. $4x^4 - 12x^3y + 5x^2y^2 + 12xy^3 - 9y^4$ and $12x^4 - 36x^3y + 11x^2y^2 + 48xy^3 - 36y^4$.

22. $x^3 - x^2y - 11xy^2 - 4y^3$ and $x^4 + x^3y - 12x^2y^2 - 30xy^3 - 8y^4$.

23. The H. C. F. of any number of expressions must be a factor of the H. C. F. of any two of these expressions. Why? Must it be the H. C. F. itself of any two of these expressions? Explain.

Find the H. C. F. of:

24. $a^4+4a^3+4a^2$, a^3b-4ab, and $a^4b+5a^3b+6a^2b$.

25. $3x^4-9x^3+6x^2$, $x^3-9x^2+26x-24$, and $x^3-8x^2+19x-12$.

26. $a\ +a^2x-2x^3$, $a\ +3a^2x+4ax^2+2x^3$, and
$2a^3+3a^2x+2ax^2-2x^3$.

II. LOWEST COMMON MULTIPLE

79. Multiples of algebraic expressions. A **multiple** of an algebraic expression is another algebraic expression that is exactly divisible by the given one; hence it contains all the prime factors of the given expression. A **common multiple** of two or more algebraic expressions is a multiple of each of these expressions.

E.g., $12a^4x^3(y^2-1)$ is a common multiple of $3a^2x^3(y+1)$ and $2a^4x(y-1)$.

The **lowest common multiple** (L. C. M.) of two or more algebraic expressions is the algebraic expression of lowest degree which is exactly divisible by each of the given expressions; hence it contains all the prime factors of each of the given expressions, but no superfluous factors.

E.g., a common multiple of $2a^3b^2x^5$ and $3a^2x^3y^4$ *must* contain the factors 2, 3, a^3, b^2, x^5, and y^4; it *may* contain other factors also, but it need not do so. Therefore $6a^3b^2x^5y^4$ is the *lowest* common multiple (L. C. M.) of $2a^3b^2x^5$ and $3a^2x^3y^4$.

So, too, the L.C.M. of $12m^3(x^2-k^2)$ and $8b^2m^2(s+t)(x-k)^2$ is $24m^3b^2(s+t)(x-k)^2(x+k)$, — show that this last expression contains all the *necessary*, but no *superfluous*, factors.

The procedure for finding the L. C. M. of two or more expressions whose prime factors are known (or easily found) may be formulated thus:

To the L. C. M. of the numerical coefficients annex all the different prime factors that occur in the given expressions, and give to each of these factors the highest exponent which that factor has in any of the given expressions.

EXERCISE LIII

Find the H. C. F. and the L. C. M. of:

1. $8\,a^2b^2$, $24\,a^4b^2c^2$, and $18\,abc^3$.
2. $15\,a^3b^4$, $-20\,a^2b^2c^2$, and $30\,ac^3$.
3. $16\,a^2b^3c$, $24\,a^3dc$, and $36\,a^4b^2d^2$.
4. $18\,a^2br^2$, $12\,p^2q^2r$, and $-54\,ab^2p^3$.
5. x^2-y^2 and $x^2+2\,xy+y^2$.
6. $21\,x^3$ and $7\,x^2(x+1)$.
7. x^2-1 and x^2+x.
8. $4\,x^2y-y$ and $2\,x^2+x$.

9. Is $12\,a^3b^4(x^2-y^2)$ a common multiple of $2\,a^2b(x-y)$ and $3\,ab^3(x-y)$? Is it their L. C. M.?

10. What factors must an expression contain in order that it may be a common multiple of two or more other expressions? that it may be their L. C. M.?

11. Are both $6\,ax^2$ and $-6\,ax^2$ multiples of $3\,x$? Explain. If a multiple of an expression has its sign reversed, does it remain a multiple of the given expression?

12. Does a change in the sign of an expression affect the *degree* of the expression? If the L. C. M. of several expressions has its sign reversed, it may still be regarded as their L. C. M. Why? (Cf. Exs. 14–15, p. 99.)

Find the L. C. M. of:

13. $a+b$, $a-b$, a^2+b^2, and a^4+b^4.
14. $3+a$, $9-a^2$, $3-a$, and $5\,a+15$.
15. x^3-y^3, x^2+xy+y^2, and x^2-xy.
16. $4\,a+4\,b$, $6\,a^2-24\,b^2$, and $a^2-3\,ab+2\,b^2$.
17. x^3+y^3, x^3y-y^4, and x^6-y^6.
18. $y^2-5\,y+6$ and $y^2-7\,y+10$.
19. $x^2-(a+b)x+ab$ and $x^2-(a-b)x-ab$.
20. $3\,s^2-7\,s+2$ and $6-s-s^2$.

HINT. $6-s-s^2=-1\,(s^2+s-6)$.

21. $c^2-4\,c+4$, $4-c^2$, and c^4-16.
22. $3\,p^2-13\,p+14$ and $13\,p-5\,p^2-6$.
23. $r^{2n}-s^{2n}$ and $(s^n-r^n)^2$.
24. $(m+n)^2-p^2$ and $(m+n+p)^2$.

25. b^3+2b^2-4b-8, $8b-12+b^2-b^3$, and $b^3+4b^2-3b-18$.

26. Find the L. C. M. of each of the sets of expressions in Exs. 19–25, p. 100.

80.* **The L. C. M. of two algebraic expressions found by means of their H. C. F.** The use of the H. C. F. in finding the L. C. M. may be shown as follows:

Let it be required to find the L. C. M. of $3x^4-x^3-x^2+x-2$ and $2x^3-3x^2-2x+3$.

By § 76 it is found that the H. C. F. of these expressions is x^2-1; they may, therefore, be written thus:

$$3x^4-x^3-x^2+x-2=(x^2-1)(3x^2-x+2),$$

and

$$2x^3-3x^2-2x+3=(x^2-1)(2x-3),$$

wherein $3x^2-x+2$ and $2x-3$ have no common factor. Hence the L. C. M. of the given expressions is

$$(x^2-1)(3x^2-x+2)(2x-3).$$

This shows that *the L. C. M. of the given expressions may be found by dividing their product by their H. C. F.*

Obviously, the L. C. M. of any other pair of expressions may be found in the same way; hence,

To find the L. C. M. of two algebraic expressions, divide either of the given expressions by their H. C. F. and multiply the other expression by the resulting quotient.

81.* **The L. C. M. of three or more expressions.** The L. C. M. of three or more algebraic expressions whose factors are not easily found, may be obtained by first finding the L. C. M. of two of the given expressions, then the L. C. M. of that result and another of the given expressions, and so on.

EXERCISE LIV

Find the L. C. M. of:

1. $x^3-6x^2+11x-6$ and $x^3-9x^2+26x-24$.

2. $x^3-5x^2-4x+20$ and $x^3+2x^2-25x-50$.

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*Articles 80 and 81, with Exercise LIV, may, if the teacher prefers, be omitted till the subject is reviewed.

3. $2y^3 - 11y^2 + 18y - 14$ and $2y^3 + 3y^2 - 10y + 14$.

4. $6a^3x - 5a^2x - 18ax - 8x$ and $6a^3b - 13a^2b - 6ab + 8b$.

5. $4x^4 - 17x^2y^2 + 4y^4$ and $2x^4 - x^3y - 3x^2y^2 - 5xy^3 - 2y^4$.

6. $2x^4 - 9x^3 + 18x^2 - 18x + 9$ and $3x^4 - 11x^3 + 17x^2 - 12x + 6$.

7. If A, B, and C stand for any three given expressions, and if M is the L. C. M. of A and B, while N is the L. C. M. of M and C, show that N is the L. C. M. of A, B, and C; that is, show that N contains all the factors necessary in such a multiple, and no superfluous factors.

Find the L. C. M. of:

8. $s^4 - 2s^3 + s^2 - 1$, $s^4 - s^2 + 2s - 1$, and $s^4 - 3s^2 + 1$.

9. $c^3 + 3c^2 - 6c - 8$, $c^3 - 2c^2 - c + 2$, and $c^2 + c - 6$.

10. $x^2 - 4a^2$, $x^3 + 2ax^2 + 4a^2x + 8a^3$, and $x^3 - 2ax^2 + 4a^2x - 8a^3$.

11. $a^3 + 7a^2 + 14a + 8$, $a^3 + 3a^2 - 6a - 8$, and $a^3 + a^2 - 10a + 8$.

12. $k^3 - 9k^2 + 23k - 15$, $k^3 + k^2 - 17k + 15$, and $k^3 + 7k^2 + 7k - 15$.

CHAPTER VIII

ALGEBRAIC FRACTIONS

82. Definitions. An **algebraic fraction** is an indicated division in which the divisor is an algebraic expression: the dividend may be either an algebraic or a numerical expression. (Cf. § 8.)

Here, as in arithmetic, the fraction $A \div B$ is usually written in the form $\frac{A}{B}$ or A/B; A and B are called the **terms** of the fraction, A being the **numerator** and B the **denominator.**

If A is exactly divisible by B, then A/B is, in reality, an integral expression, but is written in the **form of a fraction.**

E.g., $\frac{3ax}{y^2+m}$, $\frac{5}{a-2b}$, and $\frac{s}{t^2}$ are algebraic fractions; while $\frac{ab-a^2}{a}$, $\frac{m-2n}{1}$, and $\frac{a^2-x^2}{a-x}$ are integral expressions written in fractional form.

If both terms of a fraction involve the same letter, and if the numerator is not of lower degree than the denominator (in this letter), then the fraction is said to be **improper**; otherwise it is **proper.** An expression that is partly integral and partly fractional is called a **mixed expression.**

E.g., $\frac{x^2-2x+4}{x-1}$ and $\frac{a+5}{a}$ are improper fractions, and $4x-3+\frac{2a}{x-1}$ is a mixed expression.

83. Operations with fractions. The reduction of fractions, and the various operations with fractions (addition, subtraction, etc.), are essentially the same in algebra as in arithmetic.

Thus, if $\frac{A}{B}$ and $\frac{C}{D}$ are any two fractions whatever, then

$$(1)\quad \frac{A}{B}\cdot\frac{C}{D}=\frac{AC}{BD}. \qquad (2)\quad \frac{A}{B}\div\frac{C}{D}=\frac{A}{B}\cdot\frac{D}{C}.$$

These formulas state the rules for finding the product and quotient, respectively, of two fractions ; the pupil may translate each formula into verbal language.

(i) The proof of (1) follows directly from the definition of a fraction (cf. §§ 82, 8).

Thus, let $\frac{A}{B}=x$ and $\frac{C}{D}=y$,

then $A=x\cdot B$ and $C=y\cdot D$, [§§ 82, 8

hence $A\cdot C=xB\cdot yD=xy\cdot BD$, [Ax. 3

and therefore $\frac{AC}{BD}=xy$ [Ax. 4

i.e., $=\frac{A}{B}\cdot\frac{C}{D}$, [since $A/B=x$ and $C/D=y$

i.e., $\frac{A}{B}\cdot\frac{C}{D}=\frac{AC}{BD}$,

which was to be proved.

(ii) To prove (2) above, let $\frac{A}{B}\div\frac{C}{D}=t$,

then $\frac{A}{B}=t\cdot\frac{C}{D}$, [§§ 82, 8

hence $\frac{A}{B}\cdot\frac{D}{C}=t\cdot\frac{C}{D}\cdot\frac{D}{C}$ [Ax. 3

$=t\cdot\left(\frac{C}{D}\cdot\frac{D}{C}\right)=t$, [(i) above

and therefore $\frac{A}{B}\div\frac{C}{D}=\frac{A}{B}\cdot\frac{D}{C}$,

which was to be proved.

Remark. The **reciprocal** of any given number is 1 divided by that number; *e.g.*, the reciprocal of 3 is $\frac{1}{3}$.

Hence it follows from (ii) that the reciprocal of a fraction is that fraction inverted.

Note. Observe that the validity of § 42 is assumed in the proofs of (i) and (ii) above.

84. Reducing an improper fraction to a mixed expression. This change in form is made in algebra precisely as it is made in arithmetic.

E.g., just as $\frac{10}{3} = 3\frac{1}{3}$, *i.e.*, $3 + \frac{1}{3}$, so, too, *since a fraction is an indicated division*, $\dfrac{x^3 + 2x^2 + 5}{x^2 + x + 1} = x + 1 + \dfrac{4 - 2x}{x^2 + x + 1}$.

EXERCISE LV

Reduce each of the following improper fractions to an equal integral or mixed expression, and explain your work:

1. $\dfrac{a^2 - 2ab + ac}{a}$.
2. $\dfrac{3x^2 + 9x + 2}{3x}$.
3. $\dfrac{2x^2 + 4ax + 2a^2}{x + a}$.
4. $\dfrac{6s^2 - 13s - 5}{2s - 1.}$
5. $\dfrac{k^4 + 16}{k + 2}$.
6. $\dfrac{a^4 + a^2 + 1}{a + 1}$.
7. $\dfrac{x^3 + 7x^2 - 5}{x^2 - 1}$.
8. $\dfrac{x^5 - x^3 - 2x^2 - 2x - 1}{x^2 - x - 1}$.
9. $\dfrac{6 + 6c - 5c^2 - 2c^3 + c^4}{c^2 - 3}$.
10. $\dfrac{8x^3 - 10x^2 - 3x + 5}{4x^2 - 3}$.
11. $\dfrac{3x^6 + 2x - 5}{x^3 + 2x + 1}$.
12. $\dfrac{15v^4 - 13v^2 - 8v - 1}{3v^2 + 3v + 1}$.
13. $\dfrac{7k^5 - 1}{k^3 + k + 1}$.
14. $\dfrac{18x^4 - x^3 - 2x^2 - 7}{x^3 - 3x + 1}$.

15. Is $\dfrac{a^3 - 2a + 1}{5a^3 - 8a + 3}$ a proper or an improper fraction? Why?

16. Write the reciprocal of 11; of $-a$; of $-\frac{3}{4}$; of each fraction in Exs. 1–5 (cf. Remark, § 83).

85. Reducing fractions to lowest terms. In algebra, as in arithmetic, a fraction is said to be in its **lowest terms** when its numerator and denominator have no common factor.

Hence, to reduce a fraction to its lowest terms, *divide both numerator and denominator by their H. C. F.* Instead of dividing at once by the H. C. F., we may, of course, divide

by *any* common factor, then by another, etc., until *all* common factors are divided out. Multiplying or dividing both terms of a fraction by any given number leaves the value of the fraction unchanged; for, whatever the algebraic expressions represented by A, B, and m,

$$\frac{Am}{Bm} = \frac{A}{B} \cdot \frac{m}{m} \qquad [\S\ 83\ (i)$$

$$= \frac{A}{B}, \qquad [\text{since } m/m = 1$$

which was to be proved.

E.g., $\dfrac{3\,ax^2}{4\,bxy} = \dfrac{3\,ax}{4\,by}$, which is in its lowest terms; so, too, $\dfrac{x^2-1}{x^2-2\,x+1} = \dfrac{(x+1)(x-1)}{(x-1)(x-1)} = \dfrac{x+1}{x-1}$, which is in its lowest terms.

EXERCISE LVI

Reduce each of the following fractions to its lowest terms:

1. $\dfrac{a^2-ab}{a^2-b^2}$. **4.** $\dfrac{m^2+2\,mn+n^2}{m^3+n^3}$. **7.** $\dfrac{x^3+y^3}{x^4+x^2y^2+y^4}$.

2. $\dfrac{34\,a^2b^2c^4}{51\,a^4b^2c}$. **5.** $\dfrac{2\,x^2+3\,x+1}{x^2+5\,x+4}$. **8.** $\dfrac{3\,a^2-2\,a-1}{1+a-a^3-a^2}$.

3. $\dfrac{c^2-d^2}{(c-d)^2}$. **6.** $\dfrac{(r-q)^2-s^2}{(r-q-s)^3}$. **9.** $\dfrac{a^4-a^2-20}{a^4-9\,a^2+20}$.

10. May equal *factors* be canceled from the numerator and denominator of a fraction? May equal *parts* (or factors of parts) be thus canceled? Is $\dfrac{3\,a+x}{5\,bc+x}$ equal to $\dfrac{3\,a}{5\,bc}$? Is $\dfrac{2\,m+xy}{6\,x-5\,n^2}$ equal to $\dfrac{m+xy}{3\,x-5\,n^2}$? Explain fully.

Reduce the following fractions to their lowest terms, and check your work by § 25:

11. $\dfrac{s^2-t^2}{s^5t-t^6}$. **13.** $\dfrac{xy-zy-x+z}{y^2-1}$.

12. $\dfrac{c^3-17\,c^2+72\,c}{c^2(cd+16-2\,c-8\,d)}$. **14.** $\dfrac{50-40\,m+8\,m^2}{125-8\,m^3}$.

15. $\dfrac{y^{2p}-7y^p+12}{y^{2p}-8y^p+16}$.

16. $\dfrac{x-a}{(a-x)(x-b)}$.

17. $\dfrac{(s-2)(s-3)}{(2-s)(3-s)(s-4)}$.

18. $\dfrac{p^2-11p+24}{56+p-p^2}$.

19. $\dfrac{x^3+y^3}{y^4-x^4}$.

20. $\dfrac{x^4+4y^4}{x^2-2xy+2y^2}$.

21. $\dfrac{k^{4m}-1}{1+k^{6m}}$.

22. $\dfrac{(x-5)(x+2)}{x^3-7x^2+2x+40}$.

23. $\dfrac{3a^2+8a-3}{3a^3+17a^2+21a-9}$.

24. $\dfrac{10x^3+20x^2-x-2}{3x^3+6x^2+21x+42}$.

86. Reducing fractions to equal fractions having given denominators. Since multiplying both terms of a fraction by the same number does not change its value (§ 85), therefore any given fraction may be reduced to an equal fraction whose denominator is any desired multiple of the given denominator.

E.g., to reduce $\dfrac{3a}{4x^2}$ to an equal fraction whose denominator shall be $12cx^2y$, multiply both terms of the given fraction by $12cx^2y \div 4x^2$, *i.e.*, by $3cy$.

EXERCISE LVII

Find the required part in each of the following equations:

1. $\dfrac{b^2}{4}=\dfrac{?}{12}$.

2. $\dfrac{3ab}{4}=\dfrac{?}{12t}$.

3. $\dfrac{2cd}{5t}=\dfrac{16c^2d^3}{?}$.

4. $\dfrac{3c-1}{2d^2}=\dfrac{?}{8cd^4}$.

5. $\dfrac{4x}{1}=\dfrac{?}{7x^2-5}$.

6. $\dfrac{a-b}{a+b}=\dfrac{a^2-b^2}{?}$.

7. $\dfrac{m-2}{9m}=\dfrac{(m-2)^2}{?}$.

8. $\dfrac{-r}{r^2+3}=\dfrac{?}{r^4+10r^2+21}$.

9. $\dfrac{2u-v}{u+2v}=\dfrac{?}{3u^2+5uv-2v^2}$.

10. $\dfrac{u^2-uv+v^2}{7u^3-1}=\dfrac{2(u^3+v^3)}{?}$.

11. $\dfrac{3m-8}{2x-5}=\dfrac{?}{-2x+5}$.

12. $\dfrac{3m-8}{2(2x-5)}=\dfrac{?}{-6x(5-2x)}$.

13. If the denominator of a fraction is multiplied by any given expression, what must be done to the numerator in order to preserve the value of the fraction?

14. Change $\frac{3p+2}{p^2-9}$ to an equal fraction whose numerator is $-3p-2$; to one whose numerator is $3p^3+5p^2+2p$; to one whose denominator is $9p-p^3$; to one whose denominator is $3p^3+p^2-27p-9$.

87. Reduction of fractions to common denominators. To reduce any given fractions to equal fractions having a common denominator it is necessary only (1) to choose some common multiple (§ 79) of the denominators of the given fractions as the new denominator, (2) to divide this common multiple by the denominators of the given fractions in turn, and (3) to multiply both terms of the given fractions by the respective quotients (cf. § 86).

E.g., to reduce $\frac{3h}{2ax}$ and $\frac{bn}{3x^2}$ to equal fractions having a common denominator, we choose $6ax^2$ as the new denominator, and find, by § 86, that

$$\frac{3h}{2ax}=\frac{9hx}{6ax^2} \text{ and } \frac{bn}{3x^2}=\frac{2abn}{6ax^2}.$$

The lowest possible common denominator is, of course, the L. C. M. of the given denominators.

EXERCISE LVIII

Reduce the following to equal fractions having the lowest possible common denominator:

1. $\frac{l}{m}, \frac{-3l}{m^4}$, and $\frac{l^3}{5m^3}$.

2. $\frac{3a+1}{4}$ and $\frac{3x+4}{6}$.

3. $\frac{9-3a}{16b}$ and $\frac{3+5x}{20b^2}$.

4. $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$.

5. $\frac{-c}{x-1}, \frac{3c^2}{2x}$, and $\frac{c^2}{x^2}$.

6. $\frac{3}{-5x^2}, \frac{5y^2}{2x}$, and $\frac{x+y}{x-y}$.

7. $\dfrac{x-a}{x^2+ax}$ and $\dfrac{x+a}{2\,ax^2-2\,a^2x}$.

8. $\dfrac{3}{(m-1)(m-2)}$ and $\dfrac{5}{m-3}$.

9. $\dfrac{x+y}{x^3-y^3}$ and $\dfrac{x+y}{x^2+xy+y^2}$.

10. $\dfrac{x-y}{x+y}$, $\dfrac{x+y}{x-y}$, and $\dfrac{2}{x^5-xy^4}$.

11. $\dfrac{1}{2-m}$, $\dfrac{5}{m+2}$, and $\dfrac{2}{m^2-4}$.

HINT. First multiply both terms of $\dfrac{1}{2-m}$ by -1, so as to arrange the denominators in the same order.

12. $\dfrac{m}{m-n}$, $\dfrac{n}{n^2-m^2}$, and $\dfrac{p}{m+n}$.

13. $\dfrac{1}{1+x}$, $\dfrac{2}{1-x^2}$, and $\dfrac{3}{x^3-1}$.

14. $\dfrac{6\,x}{10+3\,x-x^2}$, $\dfrac{3\,a}{x^2-8\,x+15}$, and $\dfrac{3\,a-6\,x}{x^2-2\,x-15}$.

15. $\dfrac{a}{5-a}$, $\dfrac{a-2}{a^2-8\,a+15}$, and $\dfrac{a+1}{a^2-6\,a+5}$.

16. $\dfrac{2}{(x-1)(x-3)}$ and $\dfrac{7}{(x-8)(3-x)}$.

HINT. $\dfrac{7}{(x-8)(3-x)}$ equals $\dfrac{-7}{(x-8)(x-3)}$. Cf. Hint, Ex. 11.

17. Show that $\dfrac{c}{(c-1)(2-c)}=\dfrac{-c}{(c-1)(c-2)}$.

18. Show that $\dfrac{2-x}{(5-x)(x-3)}=\dfrac{x-2}{(x-5)(x-3)}$.

Reduce to equal fractions with the lowest common denominator:

19. $\dfrac{2}{(x-7)(x-2)}$, $\dfrac{5}{(2-x)(x-4)}$, and $\dfrac{9}{(4-x)(x-7)}$ (cf. Ex. 16).

20. $\dfrac{1}{(v-2)(v-5)}$, $\dfrac{2}{(v-5)(3-v)}$, and $\dfrac{-3}{(v-3)(7-v)}$.

21. $\dfrac{a+5}{a^2-4\,a+3}$, $\dfrac{a-2}{8\,a-a^2-15}$, and $\dfrac{a+1}{6\,a-5-a^2}$.

88. Addition and subtraction of fractions. From § 38 it follows that

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \text{ and } \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c};$$

that is, in algebra, as in arithmetic, *the sum (or difference) of two given fractions which have a common denominator is a fraction whose numerator is the sum (or difference) of the given numerators, and whose denominator is the common denominator of the given fractions.*

E.g., $\frac{m^2}{3ax}+\frac{2h}{3ax}=\frac{m^2+2h}{3ax}$, and $\frac{2a}{5(x-1)}-\frac{b^2c}{5(x-1)}=\frac{2a-b^2c}{5(x-1)}$.

If the given fractions have *unlike* denominators, they must, of course, be reduced to equal fractions having a common denominator (§ 87) before they can be added or subtracted.

Ex. 1. Find the sum of $\frac{3}{x-2}$ and $\frac{7}{x+1}$.

SOLUTION. The L. C. M. of the denominators is $(x-2)(x+1)$; and, by § 87,

$$\frac{3}{x-2}=\frac{3(x+1)}{(x-2)(x+1)}=\frac{3x+3}{(x-2)(x+1)},$$

and

$$\frac{7}{x+1}=\frac{7(x-2)}{(x+1)(x-2)}=\frac{7x-14}{(x+1)(x-2)};$$

$$\therefore \quad \frac{3}{x-2}+\frac{7}{x+1}=\frac{3x+3+7x-14}{(x+1)(x-2)}=\frac{10x-11}{(x+1)(x-2)}.$$

Ex. 2. Subtract $\frac{7}{x+1}$ from $\frac{3}{x-2}$.

SOLUTION. Proceeding as in Ex. 1, we obtain

$$\frac{3}{x-2}-\frac{7}{x+1}=\frac{3x+3-(7x-14)}{(x+1)(x-2)}=\frac{-4x+17}{(x+1)(x-2)}.$$

NOTE. The minus sign before the second fraction means, of course, that *all* of this fraction is to be subtracted, hence the need of the parenthesis in the numerator of the next fraction.

EXERCISE LIX

Simplify the following expressions and check your results:

3. $\frac{a^2}{3}+\frac{b^2}{6}$.

4. $\frac{a+3}{5}+\frac{a+5}{7}$.

5. $\frac{x-1}{2}+\frac{x+3}{5}+\frac{x+7}{10}$.

6. $\frac{c+d}{d}-\frac{b}{2d}$.

7. $\dfrac{1}{x+y}+\dfrac{1}{x-y}$.

8. $\dfrac{x}{1-a^2}-\dfrac{x}{1+a^2}$.*

9. $\dfrac{1}{2x-3y}+\dfrac{x+y}{4x^2-9y^2}$.

10. $\dfrac{a-b}{ab}+\dfrac{b-c}{bc}+\dfrac{c-a}{ac}$.

11. $\dfrac{r-s}{s}+\dfrac{r+s}{s}-\dfrac{r^2-s^2}{2rs}$.*

12. $\dfrac{1}{(x-y)^2}+\dfrac{1}{x^2+4xy-5y^2}$.

13. $\dfrac{a+b-c}{a^2-(b-c)^2}-\dfrac{a-b+c}{(a-b)^2-c^2}$.*

14. $\dfrac{1}{x^2-1}-\dfrac{1}{x^2-x-2}$.

15. $\dfrac{x+7}{x^2-3x-10}-\dfrac{x+2}{x^2+2x-35}$.

16. $\dfrac{1}{2s^2-s-1}-\dfrac{3}{6s^2-s-2}$.

17. $\dfrac{a^2-ax+x^2}{a^2+ax+x^2}-\dfrac{a+x}{a-x}$.

18. $\dfrac{1+x}{1-x}+\dfrac{1-x}{1+x}+x$.

[Suggestion. $x=\dfrac{x}{1}$.]

19. $\dfrac{a}{s-a}-\dfrac{a^2}{s^2-a^2}-2s$.

20. $\dfrac{1}{x^2-7x+12}-\dfrac{1}{x^2-5x+6}$.

21. $\dfrac{1}{x+y}+\dfrac{x-y}{x^2-xy+y^2}-\dfrac{x^2-xy}{x^3+y^3}$.

22. $\dfrac{1}{s(s-t)}+\dfrac{1}{t(s+t)}-\dfrac{1}{st}$.

23. $\dfrac{2x-3c}{x-2c}-\dfrac{2x-c}{x-c}+3x$.

Remark. Since a fraction is a quotient, its sign (the sign before the fraction) is governed by the law of signs in division; hence, whatever the expressions represented by a and b,

$$-\frac{a}{b}=\frac{-a}{b}=\frac{a}{-b}.$$

Exercises in subtraction may therefore be changed into exercises in addition; the results of such exercises are often called *algebraic sums* (cf. § 16).

E.g., Ex. 23 may be written $\dfrac{2x-3c}{x-2c}+\dfrac{-2x+c}{x-c}+3x$.

24. Write Exs. 6, 8, 11, 13, 14, 15, 19–22, above, as exercises in addition.

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* Cf. Ex. 2, Note.

Write the following as exercises in addition, and find the algebraic sum in each case:

25. $\frac{1}{a}-\frac{2}{a+1}+\frac{1}{a+2}$.

26. $\frac{x}{x-1}+x-\frac{x^2}{1-x}$.

27. $\frac{1}{a+b}-\frac{1}{a-b}-\frac{2a}{b^2-a^2}$.

28. $\frac{d}{c+d}-\frac{cd}{(c+d)^2}-\frac{cd^2}{(c+d)^3}$.

29. $\frac{1}{s-1}-\frac{1}{2(s+1)}$.

30. $\frac{a}{a-1}-1-\frac{1}{a(a-1)}$.

31. $\frac{1}{2x^2-x-1}-\frac{1}{3-x-2x^2}$.

32. $\frac{2b-a}{x-b}-\frac{3x(a-b)}{b^2-x^2}+\frac{b-2a}{x+b}$.

Simplify:

33. $\frac{-1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}$.

[Hint. The given expression, with the letters in alphabetical order, is

$$\frac{-1}{(a-b)(a-c)}+\frac{-1}{(b-c)(a-b)}+\frac{1}{(a-c)(b-c)}.]$$

34. $\frac{a}{(a-b)(a-c)}+\frac{b}{(b-a)(b-c)}+\frac{c}{(c-a)(c-b)}$.

35. $\frac{x-1}{(x-2)(x-3)}+\frac{2(x-2)}{(3-x)(x-1)}-\frac{x-3}{(x-1)(2-x)}$.

36. $\frac{1}{x^2-5xy+6y^2}-\frac{2}{x^2-4xy+3y^2}+\frac{1}{x^2-3xy+2y^2}$.

37. $\frac{1}{x^2-5x+6}+\frac{2}{3x-2-x^2}+\frac{3}{4x-3-x^2}$.

38. $\frac{bc}{(a-c)(a-b)}+\frac{ca}{(b-c)(b-a)}+\frac{ab}{(c-a)(c-b)}$.

89. Reducing mixed expressions to improper fractions. Since an integral expression may be written in the fractional form with the denominator 1, therefore reducing mixed expressions to improper fractions is merely a special case of addition.

E.g., $x+1+\frac{1}{x-1}=\frac{x+1}{1}+\frac{1}{x-1}$

$$=\frac{(x+1)(x-1)}{x-1}+\frac{1}{x-1}=\frac{x^2}{x-1}.$$

EXERCISE LX

By the method of § 89 simplify the following expressions:

1. $x-1+\dfrac{x^2}{x^2-1}$.

2. $x+1-\dfrac{2x}{x-1}$.

3. $3-c^2+\dfrac{c^4+6c-3}{c^2-2c+1}$.

4. $a^2-ab+b^2-\dfrac{b^3}{a+b}$.

5. $1-y-y^2-\dfrac{1-y^2}{1-y^4}$.

6. $3a-6b-\dfrac{16b^2-5c^2}{a+2b}$.

7. $x-x^2-x^3-\dfrac{x^4+x^3-x+1}{1+x+x^2}$.

8. $x^3-2x+4-\dfrac{4+4x^2+x^3}{1-2x+x^2}$.

9. $2a-3b-\dfrac{4a^2+9b^2}{2a+3b}$.

10. $1-ax-bx-\dfrac{ax+bx+ab}{1-ax}$.

11. May the numerator in the answer to Ex. 1 above be found by multiplying $x-1$ by x^2-1 and adding x^2 to the product? Explain this method fully (cf. § 84, also Ex. 20, p. 50).

12. By the method of Ex. 11, solve Exs. 2–5, and 8–10, above.

90. Product of two or more fractions. In § 83 it was shown that, whatever the expressions represented by A, B, C, and D,

$$\frac{A}{B}\cdot\frac{C}{D}=\frac{AC}{BD}.$$

This principle is easily extended to finding the product of any number of fractions;

$$e.g.,\quad \frac{A}{B}\cdot\frac{C}{D}\cdot\frac{E}{F}\cdot\frac{G}{H}=\frac{AC}{BD}\cdot\frac{E}{F}\cdot\frac{G}{H}=\frac{ACE}{BDF}\cdot\frac{G}{H}=\frac{ACEG}{BDFH}.$$

Hence, *the product of two or more fractions is a fraction whose numerator is the product of the numerators of the given fractions, and whose denominator is the product of their denominators.*

Ex. 1. Find the product of $\dfrac{a(x-1)}{3xy}$ and $\dfrac{6x}{a(x^2-1)}$.

SOLUTION

$$\frac{a(x-1)}{3xy}\cdot\frac{6x}{a(x^2-1)}=\frac{6ax(x-1)}{3axy(x^2-1)}=\frac{2}{y(x+1)}. \qquad [\S\ 85$$

REMARK. Observe that the factors 3, a, x, and $x-1$ might have been "canceled" even *before* the multiplication was actually performed. *Pupils should cancel wherever possible, and thus simplify their work.*

EXERCISE LXI

Find the following products, and simplify your results:

2. $\dfrac{abc}{xyz}\cdot\dfrac{x^2y}{ac^2}.$

3. $\dfrac{6\,xy}{8\,z}\cdot\dfrac{16\,y^2z^2}{9\,x^3}.$

4. $\dfrac{-5\,r^2s^3}{12\,tv^5}\cdot\dfrac{3\,r^2t}{10\,s^3v^3}.$

5. $\dfrac{-7\,x^2yz}{18\,xyz^2}\cdot\left(-\dfrac{6\,ky^2}{11\,x^2z}\right).$

6. $\dfrac{4\,mn^5}{1}\cdot\dfrac{-am}{m^3n^3}.$

7. $-3\,x^2yz^5\cdot\dfrac{x^2}{y}\cdot\dfrac{2}{9\,x^6y^2}.$

8. $\dfrac{a^{m+1}}{b^{m+2}}\cdot\dfrac{b^{m+1}}{a^m}\cdot\dfrac{a^2}{b^2}.$

9. $\dfrac{e^2-ef}{m^2-mn}\cdot\dfrac{m^2+mn}{ef-f^2}.$

10. $\dfrac{x-1}{x^2+2\,x+1}\cdot\dfrac{x+1}{x-1}.$

11. $\dfrac{x-1}{3\,x^2+8\,x+5}\cdot\dfrac{x^2+4\,x+3}{x^3-1}.$

12. $\dfrac{r^2-rs+s^2}{(p-q)^2}\cdot\dfrac{p^2-q^2}{r^4+r^2s^2+s^4}.$

13. $\dfrac{(a-x)^3}{x^3-y^3}\cdot\dfrac{x^2+xy+y^2}{a^2-2\,ax+x^2}.$

14. $\dfrac{(a-b)^2-1}{(a+b)^2-1}\cdot\dfrac{a+b+1}{a-b-1}.$

15. $\dfrac{s^2-t^2}{2\,st}\cdot\dfrac{s(s+2\,t)}{st+t^2}\cdot\dfrac{-3\,t^2}{s^2-st}.$

16. $\dfrac{a+5}{a-2}\cdot\dfrac{a+3}{a-5}\cdot\dfrac{a^2-7\,a+10}{2\,a^2+5\,a-3}.$

17. $\dfrac{x^3+y^3}{a^3+b^3}\cdot\dfrac{a^4+a^2b^2+b^4}{a^3-b^3}\cdot\dfrac{x^3-y^3}{x^4-2\,x^2y^2+y^4}.$

18. $\dfrac{v^2-t^2}{v^3+1}\cdot\dfrac{3}{5\,t^2v^4}=?$ May t^2 be canceled in this example? Explain (cf. Ex. 10, p. 112).

19. Simplify $\left(x+2\,y-\dfrac{5}{y}\right)\cdot\dfrac{3\,y}{a+x}$ by finding the product of each term of the multiplicand by the multiplier, and then adding the partial products (cf. § 32).

20. Simplify $\left(x+2\,y-\dfrac{5}{y}\right)\cdot\dfrac{3\,y}{a+x}$ by first reducing the multiplicand to an improper fraction.

Simplify, and check your results:

21. $(x-1)^2 \cdot \dfrac{a^4-1}{x^5-1}.$ **23.** $\dfrac{a+2b}{2b-a} \cdot (a^4-8ab^3).$

22. $(5s^2-36s+7) \cdot \dfrac{1-s^2}{3s-21}.$ **24.** $\left(1-\dfrac{y^2-xy}{x^2+y^2}\right) \cdot \dfrac{x^3+xy^2}{x^2-y^2}.$

25. $\left[a+\dfrac{ab}{a-b}\right]\left[(a-b)+\dfrac{3a(a-b)}{b}\right].$

26. $\left(x-\dfrac{8x-2x^2}{x^2-9x+20}\right)\left(\dfrac{x^2-10x+25}{x^2-6x+9}\right)\left(-\dfrac{x-3}{x}\right).$

27. $\left[\dfrac{b(x^2-y^2)}{a}+x+y\right]\left[\dfrac{3a^5+3a^4bx+3a^4by}{ax-3by-3bx+ay}\right].$

28. $\left\{1+\dfrac{c}{a+b}+\dfrac{c^2}{(a+b)^2}\right\}\left\{1-\dfrac{c^2}{(a+b)^2}\right\}\dfrac{(a+b)^4}{a+b+c}.$

29. $\left(\dfrac{a}{bc}-\dfrac{b}{ac}-\dfrac{c}{ab}-\dfrac{2}{a}\right)\left(1-\dfrac{2c}{a+b+c}\right)\left(\dfrac{5a}{c-a-b}\right).$

30. What does $\left(\dfrac{r}{s}\right)^3$ mean [cf. § 9 (ii)]? Show that $\left(\dfrac{r}{s}\right)^3=\dfrac{r^3}{s^3}$, and that $\left(\dfrac{m^2n}{kp^3}\right)^5=\dfrac{(m^2n)^5}{(kp^3)^5}=\dfrac{m^{10}n^5}{k^5p^{15}}.$

31. Raise the following fractions to the powers indicated:

$$\left(\frac{c^2}{3}\right)^4;\ \left(\frac{rs^2}{2t^3v^5}\right)^2;\ \left(\frac{m-2n}{3r+s}\right)^2;\ \left(\frac{-3c}{c-2}\right)^3;\ \left(\frac{a+b-c}{a^2b^5}\right)^2;\ \left(\frac{2x^2y^3z^4}{-3mnp}\right)^5.$$

32. Write the following fractions as powers of other fractions:

$$\frac{b^2+2b+1}{a^2};\quad \frac{y^4}{16x^8};\quad \frac{s^3-3s^2t+3st^2-t^3}{64+48y+12y^2+y^3}.$$

91. Division of fractions. In algebra, as in arithmetic, *to divide by any fraction gives the same result as to multiply by the reciprocal of that fraction* [cf. 83 (ii)].

E.g., $\dfrac{a^2x}{b^2y^2}\div\dfrac{cx}{by}=\dfrac{a^2x}{b^2y^2}\cdot\dfrac{by}{cx}=\dfrac{a^2}{bcy}.$

Note. If the divisor is an integral (or mixed) expression, it should be written in fractional form (that is, as an improper fraction) before proceeding as above.

EXERCISE LXII

Simplify the following:

1. $\dfrac{6x^5y}{14a^3b^4} \div \dfrac{2x^3}{2a^2b^2}$.

2. $\dfrac{a^2-121}{a^2-4} \div \dfrac{a+11}{a+2}$.

3. $\dfrac{-s^2p^7}{q^4r^5} \div \dfrac{3sp}{-5q^2}$.

4. $\dfrac{p^2(p+q)}{rs-s^2} \div \dfrac{p^2}{r-s}$.

5. $\dfrac{x^3-a^3}{x^3+a^3} \div \dfrac{(x-a)^2}{x^2-a^2}$.

6. $\dfrac{14x^2-7x}{12x^3+24x^2} \div \dfrac{2x-1}{x^2+2x}$.

7. $60p^2r^3s^4 \div \dfrac{5p^2r}{-8s}$.

8. $(d^2-3d-10) \div \dfrac{d-5}{d+7}$.

9. $\dfrac{a^4-b^4}{a^4+a^2b^2+b^4} \div \dfrac{(a-b)^6}{a^6-b^6}$.

10. $\dfrac{5l^2+3l-2}{l^2+8l+7} \div (5l-2)$.

11. $\left(a+\dfrac{3x^2}{a}\right)\left(\dfrac{a^2}{3x^2}-1\right) \div \dfrac{a}{x^2}$.

12. $\dfrac{(a-b)^2-9}{(a+b)^2-9} \div \dfrac{a-b+3}{a+b+3}$.

13. $\dfrac{t^4+4v^4}{-x^3} \div (t^2+2tv+2v^2)$.

14. $\dfrac{v^2-25}{v^2+v+1} \div \dfrac{5-v}{v^3-1}$.

15. $\dfrac{x^2-1}{10+3x-x^2} \div \dfrac{x^2-12x+35}{x^2+3x+2}$.

16. $\left(1-\dfrac{2cd}{c^2+d^2}\right) \div \dfrac{d-c}{3}$.

17. $\left(1+p-\dfrac{3p^2}{1-p}\right) \div \dfrac{2p+1}{p^3}$.

18. $\dfrac{r^5-1}{r^3-9r} \div \dfrac{1-r^2}{3-r}$.

19. $\dfrac{(x+3)^2}{x^2-5x-36} \div \left(x+3+\dfrac{2x+6}{x-9}\right)$.

20. $\dfrac{2r^2-21r-11}{2r^2-13r-7} \div \left(2-\dfrac{r-3}{r-7}\right)$.

21. $\dfrac{x^3-6x^2+36x}{x^2-49} \div \dfrac{x^4+216x}{x^2-x-42}$.

22. $\dfrac{5m^6n-5n^7}{m^2n+2mn^2+n^3} \div \left(-\dfrac{m^2-mn+n^2}{m+n}\right)$.

23. $\dfrac{2x^2+13x+15}{4x^2-9} \div \dfrac{2x^2+11x+5}{4x^2-1}$.

24. $\dfrac{x^4-17x^2+16}{9a^4-34a^2+25} \div \left(-\dfrac{x^2-3x-4}{3a^2+8a+5}\right)$.

25. $\dfrac{p^4-q^4}{(p-q)^2} \div \dfrac{p^2+pq}{p-q} + \dfrac{p^2}{p^2+q^2} \cdot \dfrac{p^4-2p^2q^2+q^4}{p^2-q^2}$ (cf. § 10).

26. $\left(\dfrac{x^2-16y^2}{p^2-pq}\right)^2 \div \left(\dfrac{x-4y}{p-q}\right)^3 \cdot \dfrac{p^2x-4py-4p^2y+px}{x^2+6xy+8y^2}$.

92. Complex fractions. In algebra, as in arithmetic, a fraction whose numerator or denominator, or both, are themselves fractional expressions, is called a **complex fraction.**

E.g., $\dfrac{\frac{1}{a}-a}{1+a}$ and $\dfrac{x-\frac{1}{x}}{x+2+\frac{1}{x}}$ are complex fractions.

Since a complex fraction is merely an indicated quotient, it may be simplified by means of §§ 89 and 91.

$$E.g., \quad \frac{x-\frac{1}{x}}{x+2+\frac{1}{x}} = \frac{\frac{x^2-1}{x}}{\frac{x^2+2x+1}{x}} = \frac{x^2-1}{x} \cdot \frac{x}{x^2+2x+1} = \frac{x-1}{x+1}.$$

In many cases, however, a complex fraction is most easily simplified by first multiplying both its terms by the L. C. M. of their own denominators.

Thus, in the above example, multiplying both terms by x gives $\dfrac{x^2-1}{x^2+2x+1}$, which (by § 85) equals $\dfrac{x-1}{x+1}$, as before.

EXERCISE LXIII

Simplify the following expressions:

1. $\dfrac{\frac{s^2}{s-p}}{\frac{s+p}{p^3}}$.

2. $\dfrac{\frac{c+d}{c-d}}{\frac{c^2-d^2}{c^2+d^2}}$.

3. $\dfrac{\frac{(m-n)^2}{m^2}}{\frac{m^3-n^3}{mn}}$.

4. $\dfrac{\frac{(x-3)(x-5)}{x-7}}{\frac{x-5}{(x-3)(x-7)}}$.

5. $\dfrac{\frac{r}{s}+\frac{t}{s}}{\frac{t}{v}-\frac{r}{s}}$.

6. $\dfrac{\frac{1}{e}-\frac{1}{f}}{\frac{1}{e^2}-\frac{1}{f^2}}$.

7. $\dfrac{a+\dfrac{b}{c}}{1-\dfrac{a^2c^2}{b^2}}$.

8. $\dfrac{x+\dfrac{6}{x-5}}{\dfrac{x-2}{x}}$.

9. $\dfrac{1+\dfrac{x^2}{y^2}}{\dfrac{x^2}{y^2}-\dfrac{y^2}{x^2}}$.

10. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}{\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}}$.

11. $\dfrac{3-\dfrac{a-c}{a-b}}{\dfrac{(a-b)^2}{2a-3b+c}}$.

12. $\dfrac{1+\dfrac{1}{r^2}+\dfrac{1}{r^4}}{1+\dfrac{1}{r}+\dfrac{1}{r^2}}$.

13. $\dfrac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}+\dfrac{a}{a+b}}$.

14. $\dfrac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$.

15. $\dfrac{3}{1+\dfrac{4}{1+\frac{1}{2}}}$.*

16. $\dfrac{x}{1+\dfrac{x}{1+x}}$.

17. $\dfrac{1}{a-\dfrac{a-1}{a+\dfrac{1}{a+1}}}$.

18. $\dfrac{x^2+xy+y^2}{x^2-y^2+\dfrac{3xy^2}{x-\dfrac{y^2}{x}}}$.

19. $\dfrac{s+t}{s+t+\dfrac{s}{s-t+\dfrac{t^2}{s+t}}}$.

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* To simplify such an expression we begin at the *end* and work backward. The first step here is to add $\frac{1}{2}$ to 1, then divide 4 by this sum, then add this quotient to 1, and finally divide 3 by this sum, obtaining the result $\frac{9}{11}$.

CHAPTER IX

SIMPLE EQUATIONS

93 Introductory remarks and definitions. As we have already seen in Chapter V, an algebraic problem states a relation between numbers whose values are known (called **known numbers**), and others whose values are at first unknown (called **unknown numbers**). It is by means of this relation, translated into an equation, that we can find the values of the unknown numbers.

Besides the numerals 1, 2, 3, $\cdots$ the letters a, b, c, $\cdots$ are often used to represent known numbers; unknown numbers are usually (though not *necessarily*) represented by the later letters of the alphabet, such as x, y, and z.

A **literal equation** is one in which one or more of the *known* numbers are represented by letters; while in a **numerical equation** *all* known numbers are represented by the numerals 1, 2, 3, etc. An **integral equation** is one whose members are integral in the unknown numbers (cf. § 34); known numbers may appear as divisors and the equation still be integral.

By the **degree** of an integral algebraic equation is meant the highest number of *unknown* factors which it contains in any one term.

E.g., of the equations (1) $4x - 5y = 10$, (2) $\frac{3x}{a} - 8 = \frac{5}{b}$, (3) $4a^2 = 2 + x$, (4) $3x^2 - 9a^3 = 4y^2$, and (5) $\frac{7}{2} - 3xy = 5y$, all are integral, (1), (2), and (3) are of the first degree, (4) and (5) are of the second degree, (2), (3), and (4) are literal, and (1) and (5) are numerical.

Equations of the first degree are usually called **simple equations**, and often also **linear equations** (cf. § 140, Note);

while equations of the second and third degrees are called **quadratic** and **cubic** equations, respectively.

94. Equations having fractional coefficients. Equations having fractional coefficients may be solved as follows:

Ex. 1. Given $\frac{5x}{6}-8=\frac{x}{2}$; to find x.

Solution. On multiplying both members of this equation by 6 (Ax. 3), it becomes $5x-48=3x$,

whence $2x=48$, [Ax. 1

and therefore $x=24$. [Ax. 4

On substitution in the given equation, 24 is found to check; it is therefore a root of that equation.

Multiplying both members of an equation by a common multiple of its denominators is usually spoken of as **clearing the equation of fractions.**

EXERCISE LXIV

Solve the following equations, checking the root in each case:

2. $\frac{x}{2}=\frac{4x}{3}-5.$

3. $\frac{2x}{3}-\frac{5x}{4}=7.$

4. $3x=\frac{x}{2}+25.$

5. $\frac{7x}{10}+\frac{5}{6}=5-\frac{2x}{15}.$

6. $s+\frac{4s}{9}-\frac{s}{6}=\frac{23}{36}.$

7. $\frac{m+8}{6}-\frac{m+4}{11}=1.$

[Cf. § 88, Ex. 2, Note.]

8. $\frac{2x-4}{5}-\frac{3x-7}{7}=2.$

9. $\frac{3v}{4}+5=91-10v.$

10. $\frac{4a}{5}-2\frac{5}{6}=-\frac{1}{10}+2\frac{1}{6}a.$

11. $\frac{x}{2}-\frac{2x}{3}+\frac{2x-3}{6}=-5.$

12. $1\frac{1}{15}=\frac{6t}{5}+\frac{6t-1}{2}-\frac{16t}{15}.$

13. $\frac{x-3}{3}+\frac{7x}{18}=\frac{4x-8}{5}.$

14. Are the above equations integral or fractional? Why?

15. How is an equation cleared of fractions? Upon what axiom does the process depend?

16. In each of Exs. 5–7 name three factors, any one of which might be used to clear the equation of fractions. How may we in each case find the *least* factor which can be used?

17. Name the degree of each equation in Exs. 22–28 below. Which of these equations are simple? quadratic? cubic?

Solve the following equations, and check as the teacher directs:

18. $\dfrac{7x-4}{5}=2x-\dfrac{4x+7}{7}.$

19. $-2x+4-(3x+2)=-\dfrac{x}{4}.$

20. $\dfrac{2(5-x)}{5}=\dfrac{x+2}{6}-\dfrac{3x-2}{7}.$

21. $\dfrac{4x}{3}=\dfrac{21x-25}{12}-\dfrac{4(3x-2)}{2}.$

22. $\dfrac{3s-15}{8}-\dfrac{2s-3}{4}=\dfrac{3}{2}s+15.$

23. $\dfrac{1.25+.5a}{.25}=\dfrac{.25a-2.375}{1.125}.$

24. $\dfrac{a-32}{2}-\dfrac{2a-3}{4}=\dfrac{15-3a}{8}.$

25. $x^2-x=6.$ (Cf. § 72.)

26. $x^3-15x^2+56x=0.$

27. $v^3+2v^2=v+2.$

28. $\dfrac{k+4}{7}\div\dfrac{4}{2k-5}=\dfrac{k^2-25}{10}.$

29. $\dfrac{z^2-2z-15}{\dfrac{z-3}{3}}=2\tfrac{1}{7}z.$

30. $\dfrac{\dfrac{2x}{3}+4}{2}=\dfrac{16\frac{1}{2}-x}{3}-\dfrac{x}{2}.$

31. $\frac{1}{3}[c^2-\frac{1}{4}(c^2-3)+6c-7]=9\tfrac{11}{12}.$

32. $\left(\dfrac{x}{2}-1\right)\left(\dfrac{x}{3}+2\right)-\left(\dfrac{x}{4}-3\right)\left(\dfrac{x}{3}+2\right)=-\dfrac{1}{12}.$

95. Equivalent equations. (i) Two equations are said to be **equivalent** if every root of either is a root of the other also. Thus, the several equations in the solution of Ex. 1, § 94, are equivalent; each has the root 24, and that only.

The method of solution used in Ex. 1, § 94, consists (1) in deducing from the given equation, by means of the axioms, a succession of *new* and simpler equations, and (2) in finding the root of the last and simplest of them all.

That the root of this last equation (24, in this case) happens to be a root of the given equation also is due to the fact that applying the axioms to equations *usually* produces equivalent equations.

(ii) Although the axioms are correct, their application to equations does not *always* result in *equivalent* equations.

E.g., given the equation $3x-4=1.$

Multiplying both members (Ax. 3) by $x-2$, we obtain

$$3x^2-10x+8=x-2.$$

Simplifying, $3x^2-11x+10=0,$

i.e., $(x-2)(3x-5)=0;$

whence (§ 72), $x=2$ or $x=\frac{5}{3};$

but 2 is *not* a root of the given equation.

The axioms must, therefore, be used with caution, and results should always be checked.

(iii) The following changes in equations will, however, always produce equivalent equations (cf. *El. Alg.* p. 143).

(1) Transposing and uniting terms (Axioms 1 and 2).

(2) Multiplying and dividing by any expression which is not zero, and which does not contain the unknown number (Axioms 3 and 4).

96. Literal equations. Literal equations in one unknown number, and of the first degree, may evidently be solved by the method already employed for numerical equations.

E.g., given the equation $\frac{x}{b}-\frac{x+2b}{a}=\frac{a}{b}-3;$ to find x.

On multiplying through by ab, to clear of fractions, the given equation becomes

$$ax-bx-2b^2=a^2-3ab. \qquad \text{[Ax. 3}$$

Hence $ax-bx=a^2-3ab+2b^2,$ [Ax. 1

i.e., $(a-b)x=a^2-3ab+2b^2;$

and, therefore, $x=\frac{a^2-3ab+2b^2}{a-b}$ [Ax. 4

$$=a-2b.$$

CHECK. On substituting $a-2b$ for x in the given equation, we obtain

$$\frac{a-2b}{b}-\frac{a-2b+2b}{a}=\frac{a}{b}-3,$$

in which the first member readily simplifies, and becomes $\frac{a}{b}-3$; hence $a-2b$ is a root of the given equation.

EXERCISE LXV

Solve, and check as the teacher directs:

1. $3cx-d=2d$.
2. $(a-b)x=3b-3a$.
3. $2ax=a^2-cx$.
4. $as+cs=a^2-c^2$.
5. $\frac{2x}{c}=\frac{3a}{7}+5$.
6. $\frac{c}{d}(y-1)=\frac{1}{d}$.
7. $z-3az=(1-3a)^2$.
8. $cx-c^2+4d^2=2dx$.
9. $\frac{cz+d}{c^2}=\frac{z+9cd}{3c^2}$.
10. $n^3x-x=n-x-e$.
11. $\frac{t}{e}-\frac{t}{f}=1$.
12. $\frac{t}{e-f}-\frac{t}{e+f}=1$
13. $\frac{z-a^2}{b}=\frac{b^2-z}{a}$.
14. $(b-c)x-(a+b)x=d-20$.
15. $\frac{2x-4a}{b}+\frac{5x}{a}=16+\frac{15b}{a}$.
16. $3d(x+3cd)=c(c^2-x)$.
17. $\frac{s-2ab}{c}-1=\frac{s-3c}{ab}$.
18. $\frac{x-4a}{2b}+\frac{x}{a}=\frac{a^2+4b^2}{ab}$.

19. Solve Ex. 1 for c; for d. Similarly, solve Exs. 2, 5, 9, and 11 for each letter in turn, and check your results.

Solve the following equations:

20. $b(c-x)+a(b-x)-b(b-x)=0$.
21. $a^3s+b^3s=3ab-3s(a^2b+ab^2)$.
22. $(a-b)(x-c)-(b-c)(x-a)=(c-a)(x-b)$.
23. $\frac{x+a}{b}+\frac{x+c}{a}+\frac{x+b}{c}=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+1$.

24. What is a literal equation? a numerical equation? To which class does $2x-13+ax=14x$ belong?

25. Each member of the equation $m^2 - 5m - 24 = -3(m-8)$ is divided by $m-8$; are the quotients equal? Why? Show that the new equation is not *equivalent* to the given equation.

97. Fractional equations. Equations containing expressions that are fractional in the unknown number (§ 34) are called **fractional equations.** The methods already employed apply to such equations also.

Ex. 1. Given the equation $\frac{3}{x} - \frac{1}{2} = \frac{5}{3x} + \frac{1}{6}$; to find the value of x.

Solution. Clearing of fractions by multiplying each member by $6x$, the L. C. M. of the denominators, we obtain

$$18 - 3x = 10 + x,$$

whence $$x = 2;$$

moreover, this value of x checks, hence 2 is a root of the given equation.

Ex. 2. Given $\frac{3}{2(x-1)} - \frac{1}{7(x+1)} = \frac{8}{x+1} - \frac{10}{7(x^2-1)}$; to find x.

Solution. On multiplying each member by $2 \cdot 7\ (x+1) \cdot (x-1)$, to clear the equation of fractions, we obtain

$$3 \cdot 7(x+1) - 2(x-1) = 8 \cdot 2 \cdot 7\ (x-1) - 20,$$

i.e., $$21x + 21 - 2x + 2 = 112x - 112 - 20,$$

whence $$x = \tfrac{5}{3},$$

which checks, and is, therefore, a root of the given equation.

Ex. 3. Given $\frac{7}{6} + \frac{x^3-1}{x^2-1} = x$; to find x.

Solution. On multiplying this equation by $6(x^2-1)$, to clear of fractions, we obtain

$$7(x^2-1) + 6(x^3-1) = 6x(x^2-1),$$

i.e., $$7x^2 - 7 + 6x^3 - 6 = 6x^3 - 6x,$$

whence $$7x^2 + 6x - 13 = 0,$$

i.e., $$(x-1)(7x+13) = 0,$$

and the roots of this equation are 1 and $-\frac{13}{7}$. [§ 72

On substituting these values of x in the given equation, it is found that $-\frac{13}{7}$ checks, but that 1 does not check; hence $-\frac{13}{7}$ is (and 1 is *not*) a root of the given equation.

NOTE. This shows that clearing an equation of fractions *may* introduce **extraneous roots,** *i.e.*, roots which do not belong to the given equation.

In this example the fraction $\frac{x^3-1}{x^2-1}$ might have been reduced to its lowest terms before clearing the equation of fractions. In that case the multiplier $6(x+1)$, instead of $6(x^2-1)$, would have sufficed to clear of fractions; the unnecessary factor $x-1$ brought in the extraneous root 1.

No extraneous roots are brought into a fractional equation unless an unnecessary factor is used in clearing of fractions (cf. *El. Alg.* § 99). Such roots, if introduced, are always discovered in checking.

EXERCISE LXVI

Solve and check the roots:

4. $\frac{x-3}{7}+\frac{x+5}{3}=\frac{x+26}{6}.$

5. $\frac{x-1}{2}+\frac{x-2}{3}=\frac{11-13x}{12}.$

6. $\frac{4}{x}-\frac{13}{16}=1+\frac{3}{8x}.$

7. $\frac{3}{x+1}-\frac{2}{x-1}=0.$

8. $\frac{y-1}{y+1}=1-\frac{1}{y}.$

9. $\frac{7}{10}-\frac{1}{4y}=\frac{3}{5y}-1.$

10. $\frac{x}{2}(2-x)-\frac{x}{4}(3-2x)=\frac{x+10}{6}.$

11. $\frac{2x^2}{x^2-1}+\frac{x}{x-1}=\frac{x}{x+1}+3.$

12. Clearing Ex. 11 of fractions, we obtain $x^2-2x-3=0$; are the roots of this equation, viz., 3 and -1, roots of the given equation also (cf. Ex. 3, Note)?

13. Define a fractional equation. Which of the equations in Exs. 4–11 are fractional? Explain.

Solve the following, and check as the teacher directs:

14. $\frac{10x+17}{18}-\frac{5x-2}{9}=\frac{12x-1}{11x-8}.$

HINT. Multiply both members by 18, and combine similar terms; then multiply both members of the resulting equation by $11x-18$.

15. $\frac{5(x+4)}{11}-\frac{7x-3}{3x+2}=\frac{3(3x+1)}{22}.$

16. $\dfrac{y}{3-y}-\dfrac{7}{8}+\dfrac{2y-3}{y+3}=\dfrac{y}{8(y+3)}$.

17. $\dfrac{z-5}{z+5}-\dfrac{z-10}{z+10}=\dfrac{z-4}{z+4}-\dfrac{z-9}{z+9}$.

HINT. Simplify each member before clearing of fractions.

18. $\dfrac{x+1}{x+2}-\dfrac{x+2}{x+3}=\dfrac{x+5}{x+6}-\dfrac{x+6}{x+7}$.

19. $\dfrac{x-1}{x-2}+\dfrac{x-7}{x-8}=\dfrac{x-5}{x-6}+\dfrac{x-3}{x-4}$.

20. $\dfrac{x^3+2}{x+1}-\dfrac{x^3-2}{x-1}=\dfrac{10-2x^3}{x^2-1}$.

21. $\dfrac{1}{2}+\dfrac{x}{2(x+1)}=\dfrac{x+5}{x+6}$.

22. $\dfrac{2s}{3}\cdot\dfrac{5s-3}{10s^2-1}=\dfrac{1}{3}$.

23. $\dfrac{1}{1-x}+\dfrac{6(1-x)}{x}=\dfrac{-x}{x-1}$.

24. $\dfrac{2x+1}{2x-1}-\dfrac{8}{4x^2-1}=\dfrac{2x-1}{2x+1}$.

25. $\dfrac{a}{x}-\dfrac{x-a^2}{bx}-\dfrac{1}{a}=0$.

26. $\dfrac{2c}{a}+\dfrac{b}{x}=\dfrac{c(a-2x)}{a(2-x)}$.

27. $\dfrac{x^2+ax}{x^2-cx+ax-ac}=\dfrac{x-a}{c-x}$.

28. $\dfrac{3v}{v+1}-\dfrac{15}{3v^2+v-2}=\dfrac{10}{3v-2}-5$.

29. $\dfrac{2}{m-5}-\dfrac{5m}{3m+2}=\dfrac{m+29}{(m-5)(3m+2)}-3$.

30. $\dfrac{x+7a}{x+6a}+\dfrac{x-a}{x-3a}=\dfrac{x+7a}{x+a}-\dfrac{a-x}{2a+x}$.

31. $\dfrac{1}{a(b-x)}+\dfrac{1}{b(c-x)}-\dfrac{1}{a(c-x)}=0$.

32. $\dfrac{17+\dfrac{3}{x}}{3}+\dfrac{1+\dfrac{18}{x}}{5}=\dfrac{\dfrac{21}{x}-1}{9}+\dfrac{\dfrac{100}{x}+\dfrac{5}{3}}{15}$.

33. If C represents the circumference of a circle whose radius is R, then $\dfrac{C}{2R}=\pi$ (cf. Ex. 37, p. 67); solve this equation for C; for R. Taking $\pi=3\frac{1}{7}$, find the value of R when $C=56$.

Solve each of the following equations for each letter it contains:

34. $v=\frac{s}{t}$.

35. $at=v$.

36. $D=\frac{W}{V}$.

37. $W=-Rs$.

38. $F=\frac{Mv}{t}$.

39. $v=u-gt$.

40. $\frac{5}{9}(F-32)=C$.

41. $s=\frac{1}{2}g(2t-1)$.

42. $\frac{1}{p}+\frac{1}{p'}=\frac{1}{f}$.

PROBLEMS

1. Three fourths of a certain number exceeds $\frac{1}{8}$ of it by 25. What is the number?

2. The sum of a certain number, its half, and its third is 36. Find the number.

3. If $\frac{5}{6}$ of a certain number diminished by $\frac{1}{4}$ of that number equals 3 more than $\frac{1}{3}$ of the number, what is the number?

4. The sum of two numbers is 18, and the quotient of the less divided by the greater is $\frac{1}{5}$. What are the numbers?

5. Divide the number 32 into two parts such that $\frac{1}{5}$ of the larger shall equal $\frac{1}{3}$ of the smaller.

6. Divide the number 80 into two parts such that $\frac{2}{3}$ of the smaller shall exceed $\frac{1}{4}$ of the greater by 2.

7. Divide the number 25 into two parts such that the square of the greater shall exceed the square of the smaller by 75.

8. What number must be added to each term of the fraction $\frac{7}{11}$ so that the resulting fraction shall be equal to $\frac{3}{4}$?

9. If a certain number is added to, and also subtracted from, each term of the fraction $\frac{5}{8}$, the first result exceeds the second by $\frac{1}{2}$; find the number. How many solutions has this problem?

10. B's present age is 18 years, which is $\frac{2}{3}$ of A's age; after how many years will B's age be $\frac{5}{6}$ of A's age?

11. The combined cost of a table and a chair is \$ 11, of the table and a picture, \$ 14, and the chair and the picture together cost 3 times as much as the table. What is the cost of each?

12. Divide a line 28 inches long into two parts such that the length of one part shall be $\frac{3}{4}$ that of the other.

13. A field is twice as long as it is wide, and increasing its length by 20 rods and its width by 30 rods would increase its area by 2200 square rods. What are the dimensions of this field (cf. Exs. 23–24, p. 65)?

14. An orchard has twice as many trees in a row as it has rows. By increasing the number of trees in a row by 2, and the number of rows by 3, the whole number of trees will be increased by 126. How many trees are there in the orchard?

15. An officer in forming his soldiers into a solid square, with a certain number on a side, finds that he has 49 men left over; and if he puts one more man on a side, he lacks 50 men of completing the square. How many men has he?

16. A boy was engaged at 15 cents a day to deliver a daily paper, with the added condition, however, that he was to forfeit 5 cents for every day he failed to perform this service; at the end of 60 days he received \$ 7. How many days did he serve?

17. A man was hired for 30 days on the following terms: for every day he worked he was to receive \$ 2.50 and board; for every day he was idle he was to receive nothing, and was to pay 75 cents for board. If his total earnings were \$ 49, how many days did he work?

18. The square of a certain number is diminished by 9, and the remainder is divided by 10, giving a quotient which is 3 greater than the number itself. Find the number (two solutions).

19. If a certain number is subtracted from each of the four numbers 20, 24, 16, and 27, the product of the first two remainders equals the product of the second two. What is the number?

20. Find a fraction whose numerator is greater by 3 than one half of its denominator, and whose value is $\frac{2}{3}$.

21. The numerator of a certain fraction is less by 8 than its denominator, and if each of its terms is decreased by 5, its value will be $\frac{1}{4}$; what is the fraction?

22. What principal at 4 % interest for 3 years amounts to \$ 784 (cf. Ex. 12, p. 61)? Solve the same problem if the amount is \$ 10,140.

23. I invest $6000, part at 6 %, part at 5 %, thus securing a total yearly income of $325; how large is each investment?

24. A gentleman made two investments amounting together to $4330; on one he lost 5 %, on the other he gained 12 %. If his *net* gain was $251, how large was each investment?

25. In a certain quantity of gunpowder, made up of saltpeter, sulphur, and charcoal, the saltpeter weighs 6 lb. more than $\frac{1}{2}$ of the whole, the sulphur 5 lb. less than $\frac{1}{3}$ of the whole, and the charcoal 3 lb. less than $\frac{1}{4}$ of the whole. How many pounds of each constituent does this gunpowder contain?

26. A boy bought some apples for 24 cents; had he received 4 more for the same sum, the cost of each would have been 1 cent less. How many did he buy?

27. Knowing the time consumed by an automobile in making a run of a given number of miles, how can you find the average speed? How, from the distance and the rate, can you find the time? How, from the rate and the time, can you find the distance? Illustrate your answers (cf. Exs. 15–16, p. 61).

28. A tourist ascends a certain mountain at an average rate of $1\frac{1}{4}$ miles an hour, and descends by the same path at an average rate of $4\frac{1}{2}$ miles an hour. If it takes him $6\frac{2}{3}$ hours to make the round trip, how long is the path (cf. Exs. 35–36, p. 67)?

29. A north-bound and a south-bound train leave Chicago at the same time, the former running 2 miles an hour faster than the latter. If at the end of $1\frac{1}{2}$ hours the trains are 141 miles apart, find the rate of each.

30. In running 180 miles, a freight train whose rate is $\frac{3}{5}$ that of an express train takes 2 hours and 24 minutes longer than the express train. Find the rate of each.

31. If the freight train of Ex. 30 requires 6 hours longer than the express train to make the run between Buffalo and New York, how far apart are these two cities?

32. An express train whose rate is 40 miles an hour starts 1 hour and 4 minutes after a freight train and overtakes it in 1 hour and 36 minutes. Find the rate of the freight train.

33. An automobile runs 10 miles an hour faster than a bicycle, and it takes the automobile 6 hours longer to run 255 miles than it does the bicycle to run 63 miles. Find the rate of each.

How many solutions has the equation of this problem? Is each of these also a solution of the problem itself?

34. A steamer now goes 5 miles downstream in the same time that it takes to go 3 miles upstream, but if its rate each way is diminished by 4 miles an hour, its downstream rate will be twice its upstream rate. What is its present rate in each direction?

35. A steamer can go 20 miles an hour in still water. If it can go 72 miles with the current in the same time that it can go 48 miles against the current, how swift is the current?

Hint. Let $x =$ the rate of the current (in miles per hour); then $20 - x =$ the steamer's rate upstream, and $20 + x$ its rate downstream. (Why?)

36. A man rows downstream at the rate of 6 miles an hour, and returns at the rate of 3 miles an hour. How far downstream can he go and return if he has $2\frac{1}{2}$ hours at his disposal? At what rate does the stream flow?

37. At what time between 2 and 3 o'clock are the hands of a clock together?

Hint. Make drawing, or use model of clock face. Let $x =$ the number of minute spaces over which the minute hand passes after 2 o'clock before the two hands come together; then $\frac{x}{12} =$ the number of minute spaces over which the hour hand passes in the same time (why?); and $x = \frac{x}{12} + 10$. (Why?)

38. At what time are the hands of a clock together between 8 and 9? between 5 and 6? 6 and 7? 11 and 12?

39. At what time between 3 and 4 o'clock is the minute hand 15 minute spaces ahead of the hour hand?

40. At what time do the hands of a clock extend in opposite directions between 4 and 5? between 2 and 3? 7 and 8?

41. The tens' digit of a certain two-digit number is $\frac{1}{2}$ the units' digit, and if this number, increased by 27, is divided by the sum of its digits, the quotient will be $6\frac{1}{4}$. What is the number (cf. Prob. 4, p. 64)?

42. Divide 72 into four parts, such that if the first is divided by 2, the second multiplied by 2, the third increased by 2, and the fourth diminished by 2, the results will all be equal.

43. M can do a certain piece of work in 8 days, and N can do it in 12 days; in how many days can the two do it when working together (cf. Ex. 41, p. 67)?

44. Two plasterers, A and B, working together, can plaster a house of a certain size in 12 days, while A, working alone, can plaster such a house in 18 days. In how many days can B alone do the work?

45. A reservoir is fitted with three pipes, one of which can empty it in 4 hours, another in 3 hours, and the third in $1\frac{1}{2}$ hours. If the reservoir is half full, and the three pipes are opened, in what time will it be emptied?

46. The first of three outlet pipes can empty a certain cistern in 2 hr. and 40 min., the second in 1 hr. and 15 min., and the third in 2 hr. and 30 min. If the cistern is $\frac{3}{4}$ full, and all three pipes are opened, in what time will it be emptied?

47. A can do a piece of work in 6 days, and B can do it in 14 days. A, having begun this work, had later to abandon it; B took his place and finished the work in 10 days from the time it was begun by A. How many days did B work?

48. A certain number is increased by 1, and also diminished by 1; it is then found that twice the reciprocal of the second result minus 3 times the reciprocal of the first result equals $\frac{1}{4}$. What is this number? How many solutions has this problem?

49. A picture whose length lacks 2 inches of being twice its width is inclosed in a frame 4 inches wide. If the length of the frame divided by its width, plus the length of the picture divided by its width, is $3\frac{1}{3}$, what are the dimensions of the picture? How many solutions has the equation of this problem? Is each of these a solution of the problem also?

50. A gentleman invested $\frac{1}{6}$ of his capital in 4% bonds (*i.e.*, bonds yielding 4% interest per annum), $\frac{3}{4}$ of it in $3\frac{1}{2}$% bonds, and the remainder in 6% bonds, purchasing all these bonds at par. If his total annual income is \$3412.50, find his capital.

51. At what time between 9 and 10 o'clock is the hour hand 20 minute spaces in advance of the minute hand?

52. A pedestrian finds that his uphill rate of walking is 3 miles an hour, and his downhill rate 4 miles an hour. If he walked 60 miles in 17 hours, how much of this distance was uphill?

53. A wheelman and a pedestrian start at the same time for a place 54 miles distant, the former going 3 times as fast as the latter; the wheelman, after reaching the given place, returns and meets the pedestrian $6\frac{3}{4}$ hours from the time they started. At what rate does each travel?

54. In a mixture of water and listerine containing 21 ounces there are 7 ounces of listerine. How much listerine must be added to make the new mixture $\frac{3}{4}$ pure listerine?

HINT. Let $x =$ the number of ounces of listerine to be added. Then $\frac{7+x}{21+x} = \frac{3}{4}$. (Why?)

55. In an alloy of silver and copper weighing 90 oz. there are 6 oz. of copper; find how much silver must be added in order that 10 oz. of the new alloy shall contain but $\frac{2}{5}$ oz. of copper.

56. If 80 lb. of sea water contains 4 lb. of salt, how much fresh water must be added in order that 45 lb. of the new solution may contain $1\frac{2}{3}$ lb. of salt?

57. If a mixture of water and alcohol is $\frac{9}{10}$ pure alcohol, how much water must be added to one gallon of the mixture to make a new mixture $\frac{2}{3}$ pure alcohol?

58. Solve Prob. 57 if the given mixture is 80 % pure alcohol and the required mixture 50 % pure alcohol.

59. How much alcohol must be added to one gallon of a mixture 40 % pure to make a new mixture 75 % pure?

60. What fractional part of a 6 % solution of salt and water (salt water of which 6 % by weight is salt) must be allowed to evaporate in order that the remaining portion of the solution may contain 12 % of salt? that it may contain 8 % of salt? 10 %?

61. A physician having a 6 % solution of a certain kind of medicine wishes to dilute it to a $3\frac{1}{2}$ % solution. What percentage of water must he add to the present mixture?

62. If the specific gravity of brass is $8\frac{2}{5}$,* while that of iron is $7\frac{1}{2}$, and if, when immersed in water, 57 lb. of an alloy of brass and iron displaces 7 lb. of water, find the weight of each metal in the alloy.

63. If, on being immersed in water, 97 oz. of gold displaces 5 oz. of water, and 21 oz. of silver displaces 2 oz. of water, how many ounces of gold and of silver are there in an alloy of these metals which weighs 320 oz. and which displaces 22 oz. of water? Find the specific gravity of the alloy; also of gold.

98. General problems. Formulas. Interpretation of results. A problem in which the known numbers are represented by letters, instead of by arithmetical numerals, is often called a **general problem**; it includes all those particular problems which may be obtained by giving particular values to these letters. Some problems of this kind are given below.

Prob. 1. A yacht was chartered for a pleasure party of 12, the expense to be shared equally; 3 members of the proposed party being unable to go, the share of each of the others had to be increased by $2. How much was paid for the yacht? How much was each to pay under the original arrangement?

SOLUTION

Let $x =$ the number of dollars each member was to have paid, then $x + 2 =$ the number of dollars each participant *did* pay; hence $12x$ and $9(x + 2)$ each represent the number of dollars charged for the yacht;

therefore $$12x = 9(x + 2),$$

i.e., $$12x = 9x + 18,$$

and therefore $$x = 6, \text{ and } 12x = 72;$$

hence the amount each was to have paid is $6, and the rental price of the yacht is $72.

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* This means that a given volume of brass weighs $8\frac{2}{5}$ times as much as an equal volume of water.

Prob. 2. Substitute p, q, and d, for 12, 3, and 2, respectively, in Prob. 1, and solve the problem thus formed.

SOLUTION

Let $x=$ the number of dollars each member was to have paid, then $x+d=$ the number of dollars each participant *did* pay; hence px and $(p-q)\cdot(x+d)$ each represent the number of dollars charged for the yacht;

therefore $$px=(p-q)(x+d)=px+pd-qx-qd;$$

whence $$x=\frac{d(p-q)}{q}, \text{ the amount each was to pay,}$$

and $$px=p\cdot\frac{d(p-q)}{q}, \text{ the rental price of the yacht.}$$

Remark. The solutions of Probs. 1 and 2 are alike except in this: In the solution of Prob. 1 the numbers given in that problem (12, 3, and 2) have, by combining, *completely lost their identity* before the result is reached; but in the solution of Prob. 2 the given numbers (p, q, and d) *preserve their identity* to the end.

For this reason the result in Prob. 2 may be used as a **formula,** by means of which the answer to Prob. 1, or to any *like* problem, may be immediately written down.

E.g., substituting 12, 3, and 2 for p, q, and d respectively, in the solution of Prob. 2, gives the answer to Prob. 1.

The solution of Prob. 2, therefore, includes that of Prob. 1. The first problem, and all like numerical problems, are merely *particular cases* of the second, which is called a **general problem.**

Prob. 3. Divide m golf balls into two groups, in such a way that the first group shall contain n balls more than the second.

Solution. Let $x=$ the number of balls in the first group.

Then $m-x=$ the number of balls in the second group, and, therefore, by the condition of the problem,

$$x=m-x+n;$$

whence $$x=\frac{m+n}{2}, \text{ the number in the first group,}$$

and $$m-x=m-\frac{m+n}{2}=\frac{m-n}{2}, \text{ the number in the}$$ second group.

As in Prob. 2, so here, the general solution may be employed to solve any particular problem of the same kind. For example, if $m = 30$ and $n = 4$, then the two groups contain, respectively, $\frac{30+4}{2}$ and $\frac{30-4}{2}$ balls, *i.e.*, 17 and 13; while, if $m = 10$ and $n = 2$, then the two groups contain 6 and 4 balls, respectively.

If, however, $m = 10$ and $n = 14$, then the number of balls in the two groups, as given by the above solution, is $\frac{10+14}{2}$ and $\frac{10-14}{2}$, respectively, *i.e.*, 12 and -2; but since there cannot be an actual group containing -2 golf balls, therefore this last problem is impossible, and the impossibility is indicated by the negative result.

Remark. Some problems admit of negative results, and some do not, just as some problems admit of fractional results, while others do not. The nature of the *things* with which any particular problem is concerned will always make it evident whether or not fractional or negative solutions are admissible.

Prob. 4. Two boys, A and B, are running along the same road, A at the rate of a, and B at the rate of b, yd. per minute; if B is m yd. in advance of A, and if they continue running at the same rates, in how many minutes will A overtake B?

Solution. Let $x =$ the number of minutes that must elapse before A overtakes B. Then by the conditions of the problem,

$$ax = bx + m,$$

whence $x = \frac{m}{a-b}$, the number of minutes before A overtakes B.

As in the two previous problems, so here, the general solution may be employed to solve any particular problem of the same kind.

E.g., if $a = 280$, $b = 270$, and $m = 90$, then $x = \frac{90}{280-270} = 9$; *i.e.*, A will overtake B in 9 minutes.

Again, if $a = 280$, $b = 280$, and $m = 90$, then $x = \frac{90}{280-280} = \frac{90}{0}$; *i.e.*, an infinite number of minutes will elapse before A overtakes B; in other words, A will *never* overtake B. Compare § 41 (iii), also Ex. 7, p. 53.

But if $a = 280$, $b = 290$, and $m = 90$, then $x = \frac{90}{280-290} = -9$; *i.e.*, the two boys are together -9 minutes from the moment they were observed, *i.e.*, the two boys *were* together 9 minutes ago.

Let the pupil show that this interpretation of the negative result accords fully with the physical conditions of the problem.

Prob. 5. The present ages of a father and son are respectively 50 and 20 years; after how many years will the father be 4 times as old as the son?

Solution. Let $x =$ the number of years from now to the time when the father's age shall be 4 times that of the son. Then, by the conditions of the problem,

$$50 + x = 4(20 + x),$$

whence $$x = -10.$$

This means that 10 years *ago* the father's age *was* 4 times the son's.

N.B. The general problem of which Prob. 5 is a particular case, may be stated thus: The present ages of a father and son are, respectively, m and n years; after how many years will the father be p times as old as the son?

EXERCISE LXVII

6. The sum of two numbers is a, and the larger exceeds the smaller by b. What are the two numbers?

7. By substituting in the formula obtained from the solution of Prob. 6 above, solve Probs. 6 and 7, p. 64. Could Prob. 16, p. 65, be solved by means of the same formula?

8. Is Prob. 9, p. 64, a particular or a general problem? Why? Make a general problem which shall include this one as a particular case. Solve the new problem and thus find a formula by which Prob. 9, p. 64, may be solved.

9. Answer the questions in Ex. 8 above, supposing them to have been asked with regard to Probs. 4 and 12, p. 133.

10. Which of the following admit of fractional results: Probs. 14, 15, 18, p. 134; Probs. 24–26, p. 135?

11. Do any of the problems mentioned in Prob. 10 above admit of negative results? Explain.

12. By a slight change in the wording of Prob. 5 above, make an *equivalent* problem whose answer shall be positive. This answer should agree with the interpretation of the negative result given in Prob. 5.

13. By slightly changing the wording in the last particular case under Prob. 4 above, make an *equivalent* problem whose answer shall be positive.

14. What principal at $c\%$ for t years will earn i dollars simple interest? By substituting in your answer, find the principal when $c=5$, $i=270$, $t=3$; also, when $c=3\frac{1}{2}$, $i=224$, $t=8$.

15. A father is now m times as old as his son; in p years, the father's age will be n times that of the son. Find the present age of each. Also interpret your result when m is less than n. Is p positive or negative in this case?

16. Solve the equation of Prob. 2 above for d, and then find the value of d corresponding to $p=12$, $q=2$, $x=4$. May d be fractional in value? negative? Explain.

17. M can do in a days a piece of work which N can do in b days. In how many days can they do it when working together? Use this answer to solve Prob. 43, p. 137.

18. A merchant has two kinds of sugar worth, respectively, a and b cents a pound. How many pounds of each kind must he take to make a mixture of n pounds worth c cents a pound?

19. How many solutions has Prob. 18 if $a=b=c$? if $a=b$ while c differs from a? Does the answer to Prob. 18 show these facts [cf. § 41 (iii) and (iv)]?

20. An alloy of two metals is composed of m parts (by weight) of one to n parts of the other. How many pounds of each of the metals are there in a pounds of the alloy?

21. A bell made from an alloy of 5 parts (by weight) of tin to 16 of copper, weighs 4200 lb.; how many pounds of tin and of copper in the bell? How is Ex. 22 related to Ex. 21?

22. At what time between n and $n+1$ o'clock will the hands of a clock be together? By means of your answer write down the answers to Prob. 38, p. 136.

23. At what time between n and $n+1$ o'clock will the hands of a clock be pointing in opposite directions if n is less than 6? if n is greater than 6? if n equals 6? By means of your answer write down the answers to Prob. 40, p. 136.

CHAPTER X

SIMULTANEOUS SIMPLE EQUATIONS

I. TWO UNKNOWN NUMBERS

99. Indeterminate equations. A simple equation in one unknown number has but one solution (*i.e.*, one root, cf. Chapter IX), but an equation that contains two or more unknown numbers has many solutions.

E.g., in the equation $3x+2y=5$, which, when solved for y, becomes

$$y=\frac{5-3x}{2},$$

we see that if the values 1, 2, 3, -1, etc., are *assigned* to x, then y will take the corresponding values 1, $-\frac{1}{2}$, -2, 4, etc. That is, this equation is satisfied by the pairs of numbers:

$$\left.\begin{matrix}x=1\\y=1\end{matrix}\right\};\quad \left.\begin{matrix}x=2\\y=-\frac{1}{2}\end{matrix}\right\};\quad \left.\begin{matrix}x=3\\y=-2\end{matrix}\right\};\quad \left.\begin{matrix}x=-1\\y=4\end{matrix}\right\};\ \text{etc.}$$

An equation, such as the one just now considered, which has an infinite number of solutions, is for that reason called an **indeterminate equation.**

EXERCISE LXVIII

By the method of § 99 find five solutions of each of the following equations:

1. $x+3y=7$. **3.** $5x+3y=11$. **5.** $2w=5+3z$.

2. $x+y=5$. **4.** $5m+2n=15$. **6.** $v-w=7$.

7. How many solutions has each of the above equations? Why? What are such equations called?

8. How many positive integral solutions (*i.e.*, solutions in which both x and y are positive integers) has the equation $3x+2y=11$?

HINT. Solve the equation for y, and thus show that x cannot exceed 3.

9. By the method of Ex. 8 find four positive integral solutions of the equation $2x + y = 9$. How many such solutions has this equation?

10. If possible, find positive integral solutions of the equations in Exs. 1–6 above.

Show that the following have no positive integral solutions:

11. $2x - 4y = 1$. **12.** $3x + 6y = 5$. **13.** $9x + 3y = 17$.

14. Find three solutions of the equation $2x - 5y + 3z = 6$; also, three solutions of the equation $2x + 3y + 4z = 20$.

15. A farmer spent $22 in purchasing two kinds of lambs, the first kind costing him $3 each, and the second kind $5 each. How many of each kind did he buy?

HINT. Let x = the number of the first kind, and y = the number of the second kind; then $3x + 5y = 22$, where x and y are positive integers.

16. A man spends $300 for cows and sheep costing, respectively, $45 and $6 a head; how many of each does he buy?

17. In how many ways may a 19-pound package be weighed with 5-pound and 2-pound weights?

18. How many pineapples at 25 cents each, and watermelons at 15 cents each, can be purchased for $2.15?

19. Divide a line which is 100 feet long into two parts, one of which shall be a multiple of 11 feet, the other of 6 feet.

20. Find the least number which when divided by 4 gives a remainder of 3, but when divided by 5 gives a remainder of 4.

100. Simultaneous equations. Independent equations.* The equations
$$3x + 2y = 5$$
and
$$x - 2y = 7,$$
have, individually, an infinite number of solutions (cf. § 99); they also have *one* solution, viz., $x = 3$ and $y = -2$, *in common*; *i.e.*, these values of x and y satisfy each of the given equations.

A set of equations, like those above, having one or more

* If time permits, read §§ 137–140, also § 142, in connection with §§ 100–101. This plan will make the definitions, and also the operations, more concrete.

solutions in common, is usually called a **system of simultaneous equations.**

Simultaneous equations are often called **consistent equations**, while two equations which have no solution in common are called **inconsistent equations**. Thus, $x + y = 4$ and $2x + 2y = 9$ are inconsistent equations.

Two or more equations, no one of which can be derived from the others, are called **independent equations.** Thus, $3x + y = 11$ and $7x - y = 9$ are independent; but $3x + y = 11$ and $6x + 2y = 22$ are not independent, the second being obtained by multiplying each member of the first by 2.

101. Solving simultaneous equations. The solving of a system of simultaneous equations is the process of finding the solutions which these equations have in common.

Ex. 1. Solve the equations $\begin{cases} x + y = 4, & (1) \\ x - y = 2. & (2) \end{cases}$

SOLUTION. Adding these two equations, member to member (Ax. 1), gives

$$2x = 6,$$

whence $$x = 3.$$

Substituting this value of x in Eq. (1) gives

$$3 + y = 4,$$

whence $$y = 1.$$

Moreover, these numbers, viz., $x = 3$ and $y = 1$, when substituted in the given equations, check; therefore they constitute a solution of these equations.

Ex. 2. Solve the equations $\begin{cases} 3x + 2y = 26, & (1) \\ 5x + 9y = 83. & (2) \end{cases}$

SOLUTION. On multiplying both members of Eq. (1) by 5, and of Eq. (2) by 3, these equations become, respectively,

$$15x + 10y = 130, \quad (3)$$
$$15x + 27y = 249; \quad (4)$$

and (Ax. 2) subtracting Eq. (3) from Ex. (4) gives

$$17y = 119,$$

whence $$y = 7.$$

Substituting this value of y in any one of the equations containing both x and y gives

$$x = 4;$$

and since these numbers, viz., $x = 4$ and $y = 7$, check, therefore they constitute a solution of the given system of equations.

102. Elimination. Any process of deducing from two or more simultaneous equations other equations which contain fewer unknown numbers is called **elimination.** Such a process eliminates (*i.e.*, gets rid of) one or more of the unknown numbers, and thus makes the finding of a solution easier.

That particular plan of elimination which was followed in the examples given in § 101 is known as **elimination by addition and subtraction.** It is evident, moreover, that this method is applicable to any pair of such equations. The procedure may be formulated thus:

(1) *Multiply the given equations by such numbers as will make the coefficient of the letter to be eliminated the same (in absolute value) in both equations.*

(2) *Subtract or add these last two equations (according as the terms to be eliminated have like or unlike signs).*

(3) *Solve the resulting equation for the unknown number which it contains.*

(4) *Substitute that value in any one of the earlier equations and thus find the other unknown number.*

(5) *Check the results.*

NOTE. Number (2) above is permissible only because the letters have the same value in both equations (cf. § 101).

EXERCISE LXIX

Solve each of the following systems of equations and check the results:

3. $\begin{cases} 2x - y = 5, \\ 2x + 3y = 17. \end{cases}$

4. $\begin{cases} x + 2y = 9, \\ 2x + y = 15. \end{cases}$

5. $\begin{cases} x + 3y = 11, \\ 3x - 4y = 7. \end{cases}$

6. $\begin{cases} 2v + 3u = 15, \\ 4v + 9u = 33. \end{cases}$

7. $\begin{cases} 2x+3y=-1, \\ 3x+7y=6. \end{cases}$

8. $\begin{cases} 2x+5y=8, \\ -7x+10y=-17. \end{cases}$

9. $\begin{cases} 15x+77y=92, \\ 5x-3y=2. \end{cases}$

10. $\begin{cases} 6y-5x=18, \\ 12x-9y=0. \end{cases}$

11. $\begin{cases} 5s+6t=17, \\ 6s+5t=16. \end{cases}$

12. $\begin{cases} 4m-15n=32, \\ 10m-9n=-34. \end{cases}$

13. What is meant by saying that two equations are simultaneous? consistent? inconsistent? independent? Show the appropriateness of these terms.

14. If in two simultaneous equations the coefficients of the letter to be eliminated are prime to each other (cf. Ex. 11), what is the simplest multiplier for the first equation? for the second? Answer the same questions when the coefficients under consideration are *not* prime to each other (cf. Ex. 12).

Solve the following systems of equations and check the results:

15. $\begin{cases} 5p+3q=68, \\ 2p+5q=69. \end{cases}$

16. $\begin{cases} 22x-8y=50, \\ 26x+6y=175. \end{cases}$

17. $\begin{cases} 15x+14y=-45, \\ 25x-21y=-75. \end{cases}$

18. $\begin{cases} 18u+10v=59, \\ 12u-15v=28\frac{1}{2}. \end{cases}$

19. $\begin{cases} 3m-2n=7, \\ 4m-7n=-47. \end{cases}$

20. $\begin{cases} 4r+5s=-19, \\ 2r+3s=-10\frac{7}{10}. \end{cases}$

21. $\begin{cases} 35x-27y=-19, \\ 21y+40x=82. \end{cases}$

22. $\begin{cases} 28x-23y=33, \\ 63x-25y=199. \end{cases}$

103. Other methods of elimination. Besides the method of elimination described in § 102, there are several other methods that serve the same purpose; two of these, which are often useful, will now be explained.

(i) *Elimination by substitution.*

Ex. 1. Solve the system of equations $\begin{cases} 3x-4y=7, & (1) \\ 2x+3y=16. & (2) \end{cases}$

SOLUTION

From Eq. (1) $$x = \frac{7+4y}{3};$$ [§ 99

on substituting this expression for x, Eq. (2) becomes

$$2\left(\frac{7+4y}{3}\right)+3y=16, \tag{3}$$

whence (§ 94) $$y=2;$$

on substituting this value in either Eq. (1) or Eq. (2), we obtain

$$x=5.$$

Moreover, these values, viz., $x=5$ and $y=2$, check; therefore, they constitute a solution of the given system of equations.

This method of elimination is known as **elimination by substitution**; it is manifestly applicable to any such system of equations as the above.

The student may solve, by this method, the system

$$\begin{cases} 3u-4v=19, \\ 5u+2v=10, \end{cases}$$

being careful to check the result, and then write out a "rule" for applying this method to all such exercises.

(ii) *Elimination by comparison.*

Ex. 2. Solve the system of equations $$\begin{cases} 3x-4y=7, & (1) \\ 2x+3y=16. & (2) \end{cases}$$

SOLUTION

From Eq. (1), $x=\frac{7+4y}{3}$, and from Eq. (2), $x=\frac{16-3y}{2}$. Now, since x is to have the same value in each of these equations,

therefore $$\frac{7+4y}{3}=\frac{16-3y}{2}. \tag{3}$$

Solving Eq. (3) gives $y=2$,

whence, substituting this value in either of the given equations,

$$x=5.$$

Moreover, these values, viz., $x=5$ and $y=2$, check; therefore, they constitute a solution of the given system of equations.

This method of elimination is called **elimination by comparison**; it is applicable to all such systems of equations.

The student may solve, by this method, the system

$$\begin{cases} 8r + 5s = 3, \\ 12r - 7s = 48, \end{cases}$$

and then write out a "rule" for applying this method to all such exercises.

EXERCISE LXX

Solve the following systems of equations, using first elimination by substitution, and then that by comparison; observe which method is easier in the different exercises:

3. $\begin{cases} 2x + 3y = 23, \\ 5x - 2y = 10. \end{cases}$

4. $\begin{cases} 4x + y = 34, \\ 4y + x = 16. \end{cases}$

5. $\begin{cases} 2x + 7y = 34, \\ 5x + 9y = 51. \end{cases}$

6. $\begin{cases} 11x - 10y = 14, \\ 5x + 7y = 41. \end{cases}$

7. $\begin{cases} 21y + 20x = 165, \\ 77y - 30x = 295. \end{cases}$

8. $\begin{cases} 8t - 10v = 14, \\ 6t + 35v = 41. \end{cases}$

9. Using the method of elimination by comparison, solve each of the systems of equations in Exs. 3–6, p. 147.

10. Using the method of elimination by substitution, solve each of the systems of equations in Exs. 7–10, p. 148.

11. Show that elimination by comparison is merely a special case of elimination by substitution.

Solve Exs. 12–21 below; use the simplest method of elimination in each case, giving reasons for your choice of method; and check your results as the teacher directs:

12. $\begin{cases} 7x + 4y = 1, \\ 9x + 4y = 3. \end{cases}$

13. $\begin{cases} 3x + 5y = 19, \\ 5x - 4y = 7. \end{cases}$

14. $\begin{cases} x - 11y = 1, \\ 111y - 9x = 99. \end{cases}$

15. $\begin{cases} 8u - 21v = 5, \\ 6u + 14v = -26. \end{cases}$

16. $\begin{cases} 34x - 15y = 4, \\ 51x + 25y = 101. \end{cases}$

17. $\begin{cases} 39x - 15y = 93, \\ 65x + 17y = 113. \end{cases}$

18. $\begin{cases} 19s+85t=350, \\ 17s+119t=442. \end{cases}$

19. $\begin{cases} 8s-11w=0, \\ 25s-17w=139. \end{cases}$

20. $\begin{cases} 3x-11y=0, \\ 19x-19y=8. \end{cases}$

21. $\begin{cases} x+9y+42=0, \\ 25y+x+20=0. \end{cases}$

104. Equations containing fractions. The solution of simultaneous equations containing fractions is illustrated by the following examples:

Ex. 1. Given $\left\{\begin{array}{l} \dfrac{x-2}{3}-1\tfrac{3}{4}=-\dfrac{y}{4} \\ \dfrac{x}{2}+\dfrac{2y}{3}=4\tfrac{1}{2} \end{array}\right\}$; to find x and y.

Solution. On multiplying these equations by 12 and 6, respectively, and collecting terms, we obtain

$$4x+3y=29,$$

and $$3x+4y=27;$$

whence (§ 101) $$x=5 \text{ and } y=3.$$

Moreover, when substituted in the given equations, these values check; they are, therefore, the solution of those equations.

Ex. 2. Given $\left\{\begin{array}{l} \dfrac{1}{r-3s}+\dfrac{4}{r}=\dfrac{16}{r(r-3s)} \\ \dfrac{r}{3}-1-s=0. \end{array}\right\}$; to find r and s.

Solution. On multiplying these equations by $r(r-3s)$ and 3, respectively, they become $r+4(r-3s)=16,$

and $$r-3-3s=0;$$

whence (§ 101) $$r=4 \text{ and } s=\tfrac{1}{3}.$$

When substituted in the given equations, these values check; they are, therefore, the solution sought.

Ex. 3. Given $\left\{\begin{array}{l} \dfrac{1}{u}+\dfrac{1}{v}=3 \\ \dfrac{2}{u}-\dfrac{3}{v}=1 \end{array}\right\}$; to find u and v.

SOLUTION. Instead of clearing of fractions here, it is better to treat $\frac{1}{u}$ and $\frac{1}{v}$ as the unknown numbers; we may even substitute a single letter for each of these unknown fractions.

Thus, on substituting x for $\frac{1}{u}$ and y for $\frac{1}{v}$, the given equations

become $$x+y=3,$$

and $$2x-3y=1,$$

respectively. Whence $$x=2 \text{ and } y=1, \qquad [\S\ 103$$

i.e., $$\frac{1}{u}=2 \text{ and } \frac{1}{v}=1;$$

whence $$u=\tfrac{1}{2} \text{ and } v=1; \qquad [\S\ 97$$

and these values are found to check.

EXERCISE LXXI

Solve the following systems of equations, and check the results; eliminate before clearing of fractions when practicable:

4. $\begin{cases} \dfrac{x}{4}+\dfrac{y}{2}=12, \\ \dfrac{x}{4}-\dfrac{y}{2}=-2. \end{cases}$

5. $\begin{cases} \dfrac{x}{3}+\dfrac{y}{3}=7, \\ \dfrac{x}{6}+\dfrac{y}{2}=6\frac{1}{2}. \end{cases}$

6. $\begin{cases} \dfrac{x}{2}+\dfrac{y}{3}=7, \\ \dfrac{x}{3}+\dfrac{y}{4}=5. \end{cases}$

7. $\begin{cases} \dfrac{x}{5}+5z=-4, \\ \dfrac{z}{5}+5x=4. \end{cases}$

8. $\begin{cases} \dfrac{4}{x}+\dfrac{3}{y}=3, \\ \dfrac{2}{x}-\dfrac{3}{y}=1. \end{cases}$

9. $\begin{cases} \dfrac{2}{s}-\dfrac{3}{r}=5, \\ \dfrac{5}{s}-\dfrac{2}{r}=7. \end{cases}$

10. $\begin{cases} \dfrac{2r+3t}{5}+\dfrac{t+6}{7}=2, \\ \dfrac{2r-5t}{3}+\dfrac{r+7}{4}=1. \end{cases}$

11. $\begin{cases} \dfrac{h-2}{3}-\dfrac{k+5}{3}=0, \\ \dfrac{2h-7}{3}-\dfrac{13-k}{6}=10. \end{cases}$

12. $\begin{cases} \dfrac{3x+2y+6}{4x-2y}=1, \\ \dfrac{3-7y}{2x+1}=2. \end{cases}$

13. $\begin{cases} \dfrac{8}{5x+16y}=\dfrac{15}{3x-4y}, \\ 8y-2x=7. \end{cases}$

14. $\begin{cases} \dfrac{m-2}{m}-\dfrac{n-3}{n}=0, \\ \dfrac{2n-5}{2n-3}=\dfrac{3m-7}{3m+1}. \end{cases}$

15. $\begin{cases} \dfrac{3x-2y+\frac{3}{2}}{x-y}=\dfrac{16}{3}, \\ \dfrac{15+y-2x}{4x-5y-2}=5. \end{cases}$

16. $\begin{cases} 2x+y-50=0, \\ \dfrac{x}{6}+\dfrac{y}{7}=5. \end{cases}$

17. $\begin{cases} \dfrac{x}{2}+\dfrac{y}{3}-1=0, \\ \dfrac{x}{4}=\dfrac{2y}{3}+3. \end{cases}$

18. $\begin{cases} \dfrac{5}{x}+\dfrac{6}{y}=20, \\ \dfrac{6}{x}+\dfrac{5}{y}=10. \end{cases}$

19. $\begin{cases} \dfrac{3}{2v}-\dfrac{1}{w}=-3, \\ \dfrac{5}{2v}+\dfrac{3}{w}=23. \end{cases}$

20. $\begin{cases} -\dfrac{5r}{8}+\dfrac{7t}{4}=13, \\ \dfrac{11r}{12}-\dfrac{5t}{8}=12. \end{cases}$

21. $\begin{cases} \dfrac{3}{2x}+\dfrac{6}{5y}=\dfrac{17}{40}, \\ \dfrac{7}{2x}-\dfrac{4}{5y}=\dfrac{11}{120}. \end{cases}$

22. $\begin{cases} 5s-\dfrac{3}{4t}-2\frac{11}{20}=0, \\ -4s+\dfrac{15}{7t}-\dfrac{3}{7}=0. \end{cases}$

23. $\begin{cases} \dfrac{x}{2}-\frac{1}{3}(y-2)=\frac{1}{4}(x-3), \\ x-\frac{1}{2}(y-1)=\frac{1}{3}(x-2). \end{cases}$

24. $\begin{cases} \dfrac{1}{x-2}+\dfrac{2}{3-y}=0, \\ \dfrac{x-7}{6}=\dfrac{2y+11}{5.5}. \end{cases}$

25. $\begin{cases} \dfrac{.2y+.5}{1.5}=\dfrac{.49x-.7}{4.2}, \\ \dfrac{.5x-.2}{1.6}=\dfrac{41}{16}-\dfrac{1.5y-11}{8}. \end{cases}$

26. $\begin{cases} \dfrac{5x+6y+13}{4y-2x+6}=-\dfrac{3}{2}, \\ \dfrac{13}{7x-y}=\dfrac{2\frac{1}{2}}{x-y-3}. \end{cases}$

27. $\begin{cases} \dfrac{1}{4u+v}-\dfrac{2}{u}=\dfrac{1}{-u(4u+v)}, \\ \dfrac{-8}{v}=\dfrac{15}{2u-v}+\dfrac{5}{v(2u-v)}. \end{cases}$

28. $\begin{cases} v+\frac{1}{2}(3v-w-1)=\frac{1}{4}+\frac{3}{4}(w-1), \\ \frac{1}{5}(4v+3w)=\frac{1}{10}(7w+24). \end{cases}$

29. $\begin{cases} x-20-\dfrac{2y-x}{23-x}=\dfrac{2x-59}{2}, \\ y-\dfrac{3-y}{x-18}=30+\dfrac{3y-73}{3}. \end{cases}$ (Cf. Ex. 14, p. 131.)

30. $\begin{cases} \dfrac{s}{2}-3=\dfrac{5t+2s}{4-s}+\dfrac{s-3}{2}, \\ \dfrac{2t-3s}{t+1}+t=12-\dfrac{7-2t}{2}. \end{cases}$

31. $\begin{cases} 4x+\dfrac{2y-\dfrac{x}{2}}{17-3x}=\dfrac{16x+19}{4}, \\ 50-\dfrac{y-1}{\frac{5}{3}(x-2)}=8y+\dfrac{147-24y}{3}. \end{cases}$

32. $\begin{cases} \frac{3}{4}(2x+3)+\dfrac{3x+5y}{2(2x-3)}=3\frac{1}{4}+\dfrac{3x+4}{2}, \\ \dfrac{8y+7}{10}+\dfrac{6x-3y}{2(y-4)}=4+\dfrac{4y-9}{5}. \end{cases}$

105. Literal equations. Literal equations may be solved by the methods already employed in solving numerical equations.

E.g., given $\left.\begin{cases} ax+by=c \\ hx+ky=l \end{cases}\right\}$; to find x and y.

Solution. On multiplying the first of these equations by k and the second by b, they become

$$akx+bky=ck,$$

and $$bhx+bky=bl.$$

Subtracting member from member, we obtain

$$akx-bhx=ck-bl,$$

i.e., $$(ak-bh)x=ck-bl;$$

whence $$x=\frac{ck-bl}{ak-bh}.$$

If we multiply the first of the given equations by h, and the second by a, and subtract, we eliminate x, and find

$$y=\frac{ch-al}{bh-ak}.$$

Moreover, these values of x and y check, and are, therefore, a solution of the given equations.

EXERCISE LXXII

Solve the following systems of equations and check the results; eliminate without clearing of fractions where practicable:

1. $\begin{cases} x+y=c, \\ x-y=d. \end{cases}$

2. $\begin{cases} ax=by, \\ x+y=ab. \end{cases}$

3. $\begin{cases} y+az=0, \\ by+z=1. \end{cases}$

4. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=c, \\ \dfrac{1}{x}-\dfrac{1}{y}=d. \end{cases}$

5. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}-2=0, \\ bx-ay=0. \end{cases}$

6. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=2\,ab, \\ \dfrac{x}{ab}+\dfrac{y}{ab}=a+b. \end{cases}$

[Hint. Let $s=\frac{x}{a}$ and $t=\frac{y}{b}$; cf. Ex. 3, p. 152].

7. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=1, \\ \dfrac{b}{x}+\dfrac{a}{y}=1. \end{cases}$

8. $\begin{cases} ax+by=m, \\ bx+ay=n. \end{cases}$

9. $\begin{cases} x-y=a-b, \\ ax+by=a^2-b^2. \end{cases}$

10. $\begin{cases} \dfrac{x+y}{a}-\dfrac{x-y}{b}=0, \\ \dfrac{x+y}{b}-\dfrac{x-y}{a}=1. \end{cases}$

11. $\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=\dfrac{1}{c}, \\ \dfrac{c}{x}-\dfrac{a}{y}=\dfrac{1}{b}. \end{cases}$

12. $\begin{cases} \dfrac{1}{ax}+\dfrac{1}{by}=\dfrac{1}{c^2}, \\ \dfrac{1}{bx}+\dfrac{1}{cy}=\dfrac{1}{a^2}. \end{cases}$

13. $\begin{cases} (a+b)x+(a+c)y=a+b, \\ (a+c)x+(a+b)y=a+c. \end{cases}$

14. $\begin{cases} \dfrac{x+1}{y+1}=\dfrac{a+b+1}{a-b+1}, \\ x-y=2b. \end{cases}$

15. $\begin{cases} hx+ky=4h^2, \\ \dfrac{1}{x-k}+\dfrac{1}{y-h}=\dfrac{h}{k(y-h)}. \end{cases}$

16. Under what circumstances has Ex. 8 above no finite solution? Explain [cf. § 41 (iii)]. Answer this question with regard to Ex. 9 also; and with regard to Ex. 3.

II. THREE OR MORE UNKNOWN NUMBERS

106. Equations containing more than two unknown numbers. The methods already employed in the solution of systems of equations containing *two* unknown numbers (§§ 101–105) are easily extended to systems containing three or more unknown numbers.

Thus, to solve the system of equations

$$\begin{cases} x+3y-z=5, & (1)\\ 3x+6y+2z=3, & (2)\\ 2x-3y-2z=6, & (3)\end{cases}$$

we first eliminate some one of the unknown numbers, say z, between (1) and (2), then eliminate *the same* unknown number between (1) and (3); in this way we obtain two new equations, each containing the two unknown numbers x and y. On solving these two equations we find x and y, and substituting their values in (1) we find z, which completes the solution of the given system.

Ex. 1. Given

$$\begin{cases} x+3y-z=5, & (1)\\ 3x+6y+2z=3, & (2)\\ 2x-3y-3z=6; & (3)\end{cases}$$

to find x, y, and z.

SOLUTION

Adding 2 times Eq. (1) to Eq. (2), member to member, gives

$$5x+12y=13, \qquad (4)$$

and subtracting Eq. (3) from 3 times Eq. (1) gives

$$x+12y=9. \qquad (5)$$

Now subtracting Eq. (5) from Eq. (4) gives

$$4x=4,$$

whence

$$x=1.$$

On substituting this value of x in Eq. (5), we obtain $y=\frac{2}{3}$; and, with these values of x and y in Eq. (1), we obtain $z=-2$.

Moreover, these values of x, y, and z check; therefore they constitute a solution of the given system of equations.

NOTE. Had the given system consisted of four equations, containing four unknown numbers, the same method of solution would still have sufficed. For, by eliminating some one of the unknown numbers, say x, between (1) and (2), (1) and (3), and (1) and (4) in turn, we should have obtained a system of three equations containing the remaining three unknown numbers, which could then have been solved as in Ex. 1. And the values of these three unknown numbers, being substituted in any one of the given equations, would have determined the value of the remaining unknown number.

Similarly, a system consisting of five equations containing five unknown numbers can, by eliminating some one of these, be made to depend upon a system of four equations in four unknown numbers; and so in general (see also § 107).

Ex. 2. Given
$$\begin{cases} 2x-3y-2z=-1, & (1)\\ 3x+z=6, & (2)\\ x+y+z=3; & (3)\end{cases}$$
to find the values of x, y, and z.

SOLUTION. Since the second of these equations is already free from the unknown number y, therefore it is best to combine Eqs. (1) and (3) so as to eliminate y, and thus obtain another equation involving only x and z. On adding Eq. (1) to three times Eq. (3) we obtain
$$5x+z=8, \qquad (4)$$
and on subtracting Eq. (2) from Eq. (4), we obtain
$$2x=2,$$
whence
$$x=1. \qquad (5)$$
On substituting this value of x in Eq. (2), we obtain
$$z=3;$$
and on substituting these two values in Eq. (3), we obtain
$$y=-1.$$

Moreover, these values of x, y, and z, viz., 1, −1, and 3, check, and therefore constitute a solution of the given equations.

EXERCISE LXXIII

Solve each of the following systems of equations:

3. $\begin{cases} 2x+3y+4z=20,\\ 3x+4y+5z=26,\\ 3x+5y+6z=31.\end{cases}$

4. $\begin{cases} 4x-y-z=5,\\ 3x-4y+16=6z,\\ 3y+2(z-1)=x.\end{cases}$

5. $\begin{cases} 7x+3y-2z=16, \\ 2x+5y+3z=39, \\ 5x-y+5z=31. \end{cases}$

6. $\begin{cases} 5x-6y+4z=15, \\ 7x+4y-3z=19, \\ 2x+y+6z=46. \end{cases}$

7. $\begin{cases} 2x+4y+5z=19, \\ -3x+5y+7z=8, \\ 8x-3y+5z=23. \end{cases}$

8. $\begin{cases} 5x+6y-12z=5, \\ 2x-2y-6z=-1, \\ 4x-5y+3z=7\frac{1}{2}. \end{cases}$

9. $\begin{cases} y+z-86=72-5x, \\ 93-\frac{1}{2}x-\frac{1}{4}y=\frac{3}{4}y-2z, \\ \frac{1}{4}x+\frac{1}{3}y+\frac{1}{2}z=58. \end{cases}$

10. $\begin{cases} \frac{1}{2}x+\frac{1}{3}y=12-\frac{1}{6}z, \\ \frac{1}{2}y+\frac{1}{3}z=8+\frac{1}{6}x, \\ \frac{1}{2}x+\frac{1}{3}z=10. \end{cases}$

11. $\begin{cases} 2x-5y+19=0, \\ 3y-4z+7=0, \\ 2z-5x-2=0. \end{cases}$

12. $\begin{cases} \dfrac{x}{6}-\dfrac{y}{5}+\dfrac{z}{4}=3, \\ \dfrac{x}{5}-\dfrac{y}{4}+\dfrac{z}{5}=1, \\ \dfrac{x}{4}-\dfrac{y}{3}+\dfrac{z}{2}=5. \end{cases}$

13. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=6, \\ \dfrac{1}{y}+\dfrac{1}{z}=10, \\ \dfrac{1}{z}+\dfrac{1}{x}=8. \end{cases}$

14. $\begin{cases} \dfrac{3}{x}+\dfrac{2}{y}+\dfrac{1}{z}=1, \\ -\dfrac{4}{x}-\dfrac{4}{y}+\dfrac{3}{z}=10, \\ \dfrac{1}{x}+\dfrac{4}{y}=1. \end{cases}$

15. $\begin{cases} x+y-z=a, \\ x-y=2b, \\ x+z=3a+b. \end{cases}$

16. $\begin{cases} x+3y=6c, \\ y+2z=x, \\ y+z=\frac{2}{3}x-2d. \end{cases}$

17. $\begin{cases} 5+6\left(\dfrac{1}{x}+\dfrac{1}{y}\right)=0, \\ 5-2\left(\dfrac{1}{y}+\dfrac{1}{z}\right)=0, \\ 14-3\left(\dfrac{1}{x}+\dfrac{1}{z}\right)=0. \end{cases}$

18. $\begin{cases} \dfrac{xy}{x+y}=\dfrac{1}{a}, \\ \dfrac{yz}{y+z}=\dfrac{1}{b}, \\ \dfrac{xz}{x+z}=\dfrac{1}{c}. \end{cases}$

[HINT. If $\dfrac{xy}{x+y}=\dfrac{1}{a}$, then $\dfrac{x+y}{xy}=a$; *i.e.*, $\dfrac{1}{y}+\dfrac{1}{x}=a$.]

19. $\begin{cases} 2v+3x+y-z=0, \\ 3y-2x+z-4v=21, \\ 2z-3v-y+x=6, \\ v+4x+2y-3z=12. \end{cases}$

20. $\begin{cases} v+x+y=15, \\ x+y+z=18, \\ v+y+z=17, \\ v+x+z=16. \end{cases}$

HINT. Adding these equations and dividing the sum by 3 gives $v+x+y+z=22$.

21. $\begin{cases} y+z+v-x=22, \\ z+v+x-y=18, \\ v+x+y-z=14, \\ x+y+z-v=10. \end{cases}$

22. $\begin{cases} y+z-3x=2a, \\ z-v-3y=2b, \\ y+x-3z=2c, \\ 2v+2y=a-b. \end{cases}$

23. $\begin{cases} 3u+5v-2x+3z=2, \\ 2u+4x-3y-z=3, \\ u+v+z=2, \\ 6y+4v+u=2, \\ 5z+4x-7v=0. \end{cases}$

PROBLEMS

[Leading to simultaneous equations in two or more unknown numbers.]

1. Find two numbers whose difference is $\frac{1}{35}$ of their sum, and such that 5 times the smaller minus 4 times the larger is 39.

SOLUTION

Let $x =$ the larger number,
and $y =$ the smaller number.

Then, by the conditions of the problem,

$$x - y = \frac{x+y}{35},$$

and $$5y - 4x = 39.$$

Solving these equations, we obtain

$$x = 54 \text{ and } y = 51;$$

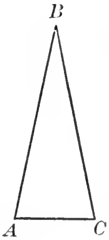
and these numbers, which constitute a solution of *the equations of the problem*, also satisfy the *problem itself*, and are, therefore, the numbers sought.

2. Find two numbers such that 3 times the greater exceeds twice the less by 29, and twice the greater exceeds 3 times the less by 1.

3. A lady purchased 20 yd. of gingham, and 50 yd. of linen, for $29; she could have purchased 30 yd. of gingham, and 20 of linen, for $16. What was the price of each material?

4. If A's money were increased by $4000, he would have twice as much as B. If B's money were increased by $5500, he would have 3 times as much as A. How much money has each?

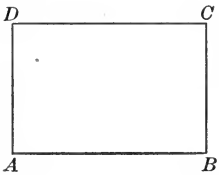
5. One eleventh of A's age is greater by 2 years than $\frac{1}{7}$ of B's, and twice B's age equals what A's age was 13 years ago. Find the present age of each.



6. ABC represents a triangle whose perimeter is 82 inches. If $AB = BC$ and $7\,BC = 17\,AC$, find the length of each side of the triangle.

7. A man having \$45 to distribute among a group of children, finds that he lacks \$1 of being able to give \$3 to each girl and \$1 to each boy, but that he has just enough to give \$2.50 to each girl and \$1.50 to each boy. How many boys and how many girls are there in this group?

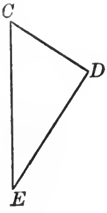
8. John said to James, "Give me 8 cents and I shall have as much as you have left." James said to John, "Give me 16 cents and I shall have 4 times as much as you have left." How much money had each?



9. $ABCD$ represents a flower bed in which $BC = \frac{2}{3}AB$. If the perimeter of the bed is 40 feet, find the length of each of its sides.

10. A pound of tea and 6 lb. of sugar together cost \$.96; if sugar were to advance 50%, and tea 10%, then 2 lb. of tea and 12 lb. of sugar would cost \$2.28. Find the present price of tea, and also of sugar.

11. A grain dealer sold to one customer 5 bushels of wheat, 2 of corn, and 3 of rye, for \$6.60; to another, 2 of wheat, 3 of corn, and 5 of rye, for \$5.80; and to another, 3 of wheat, 5 of corn, and 2 of rye, for \$5.60. What was the price per bushel of each kind of grain?



12. The perimeter of the triangle CDE is 68 in.; four times CE equals CD increased by four times DE, while twice CE equals DE increased by twice CD. How long is each side of the triangle?

13. Divide 800 into three parts such that the first, plus $\frac{1}{2}$ of the second, plus $\frac{2}{5}$ of the third, shall equal the second, plus $\frac{3}{4}$ of the first, plus $\frac{1}{4}$ of the third: each of these sums being 400.

14. Divide 90 into three parts such that $\frac{1}{2}$ of the first, plus $\frac{1}{3}$ of the second, plus $\frac{1}{4}$ of the third, shall be 30; while the first part increased by twice the second shall equal twice the third.

15. A boy spent \$4.10 for oranges, buying some at the rate of 2 for 5 cents, some at 3 for 10 cents. Later he sold all at 4 cents apiece, thereby clearing \$1.58. How many of each kind did he buy?

16. If a certain rectangular floor were 2 ft. broader and 3 ft. longer, its area would be increased by 64 sq. ft.; but if it were 3 ft. broader and 2 ft. longer, its area would be increased by 68 sq. ft. Find its length and breadth.

17. Three rectangles are equal in area; the second is 6 meters longer and 4 meters narrower than the first, and the third is 2 meters longer and 1 meter narrower than the second. What are the dimensions of each?

18. The sum of the ages of a father and son will be doubled in 25 years; the difference of their ages 20 years hence will just equal $\frac{1}{3}$ of their sum at that time. Find the present age of each.

19. A merchant sold to Mrs. A. 2 yd. of cambric, 4 of silk, and 3 of flannel, for \$5.05, and to Mrs. B., 4 yd. of cambric, 5 of flannel, and 2 of silk, for \$4.30. If 2 yd. of flannel cost 10 cents more than 2 yd. of cambric and $\frac{1}{2}$ yd. of silk combined, find the price of each per yard.

20. The tickets to a concert were 50 cents for adults and 35 cents for children. If the proceeds from the sale of 100 tickets were \$39.50, how many tickets of each kind were sold?

Solve this problem also by using but one letter to represent an unknown number.

21. Find three numbers such that the sum of the reciprocals of the first and second is $\frac{8}{15}$, the sum of the reciprocals of the first and third is $\frac{7}{18}$, and the sum of the reciprocals of the second and third is $\frac{23}{90}$.

22. The sum of the reciprocals of three numbers is 34; the reciprocal of the second minus that of the third equals 4; the sum of 3 times the reciprocal of the first and twice the reciprocal of the second is less by 1 than 5 times the reciprocal of the third. Find the three numbers.

23. In a certain two-digit number which equals 8 times the sum of its digits, the tens' digit exceeds 3 times the units' digit by 1. Find the number.

24. The sum of the digits of a two-digit number is 12, and if the digits are interchanged, the number thus formed will lack 12 of being twice the original number. What is the number?

25. The sum of the digits of a 3-digit number is 11; the double of the second digit exceeds the sum of the first and third by 1, and if the first and second digits are interchanged, the number will be diminished by 90. What is the number?

26. The third digit of a 3-digit number is as much larger than the second as the second is larger than the first; if the number is divided by the sum of its digits, the quotient is 15; and the number will be increased by 396 if the order of its digits is reversed. What is the number?

27. A capitalist invested $4000, part at 5%, part at 4%, and found that his annual income from this investment was $175. How much was invested at 5%, and how much at 4%?

Solve this problem also by using only one unknown letter.

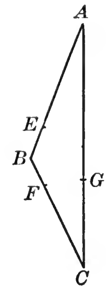
28. A capitalist invested A dollars, part at p%, part at q%, and found that his annual income from this investment was B dollars. How much was invested at p%? at q%?

Show that this problem includes Prob. 27 as a special case.

29. Divide the number N into two such parts that $1/m$ of the first part, plus $1/n$ of the second, shall exceed the first part by M.

Specialize this problem, and find the solution of the special problem by substituting in the general solution.

30. Three cities, A, B, and C, are situated at the vertices of a triangle; the distance from A to C by way of B is 50 miles, from A to B by way of C is 70 miles, and from B to C by way of A is 60 miles. How far apart are these cities? (Make diagram.)



31. In the triangle ABC, $AB = 12$ inches, $BC = 10$ inches, $BE = BF$, $FC = CG$, $AG = 4\frac{1}{3}\,BF$. If the perimeter of the triangle is 42, find AC, AE, BE, FC.

32. A quantity of water which is just sufficient to fill three jars of different sizes, will fill the smallest jar exactly 4 times; or the largest jar twice, with 4 gallons to spare; or the second jar 3 times, with 2 gallons to spare. Find the capacity of each jar.

33. Two men, A and B, rowed a certain distance, alternating in the work; A rowed at a rate sufficient to cover the entire distance in 10 hours, while B's rate would require 14. If the journey was completed in 12 hours, how long did each row?

34. Two boys, A and B, run a race of 400 yards, A giving B a start of 20 seconds and winning by 50 yards. On running this race again, A, giving B a start of 125 yards, wins by 5 seconds. What is the speed of each? Generalize this problem.

35. If A and B can do a certain piece of work in 10 days, A and C in 8 days, and B and C in 12 days, how long will it take each to do the work alone?

36. A and B together can build a wall in $5\frac{5}{11}$ days; being unable to work at the same time, A works 5 days, then B takes up the work, finishing it in 6 days more. In how many days could each have built the wall alone? Generalize this problem.

37. A man can row m miles downstream in c hours and m miles upstream in d hours; what is his rate of rowing in still water, and what is the rate of the current?

38. From the solution of Prob. 37 find the solution of the special problem in which $m = 6$, $c = 1\frac{1}{3}$, $d = 4$.

39. Two trains whose respective lengths are 1200 feet and 960 feet run on parallel tracks; when moving in opposite directions, the trains pass each other in 24 seconds; when moving in the same direction, each at the same rate as before, the faster passes the slower in $1\frac{1}{3}$ minutes. Find the rate of each train.

40. Two trains are scheduled to leave the cities A and B, m miles apart, at the same time, and to meet in h hours; but, the train from A being a hours late in starting, and running at its regular rate, the trains met k hours later than the scheduled time. What is the rate at which each train runs?

41. From the solution of Prob. 40 find the solution of the special problem in which $m = 800$, $h = 10$, $a = 1\frac{3}{5}$, and $k = \frac{7}{10}$.

42. A train was scheduled to make a certain run at a uniform speed. After traveling 2 hours it was delayed 1 hour by an accident, after which it proceeded at $\frac{10}{9}$ its usual rate and arrived $\frac{1}{2}$ hour late. Had the accident occurred 36 miles farther on, the train would have been 36 minutes late. Find the usual rate of the train and the entire distance traveled.

43. Two boats which are d miles apart will meet in a hours if they sail toward each other, and the second will overtake the first in b hours if they sail in the same direction. Find the respective rates at which these boats sail. Also discuss fully your solution, *i.e.*, interpret the results (cf. Prob. 4, p. 141).

44. Find an expression of the form $ax^2 + bx + c$ whose value is 6 when $x = 2$, 3 when $x = -1$, and 10 when $x = 4$.

HINT. $4a + 2b + c$ is the value of $ax^2 + bx + c$ when $x = 2$; therefore, $4a + 2b + c = 6$, etc.

45. Find an expression of the form $ax^2 + bx + c$ whose value is 7 when $x = 3$, 9 when $x = -1$, and 17 when $x = -5$.

46. Find an expression of the form $ax^3 + bx^2 + cx + d$ which equals -16 when $x = -1$, -4 when $x = 1$, -43 when $x = -2$, and -100 when $x = -3$.

47. Of three alloys, the first contains 35 parts of silver, to 5 of copper, to 4 of tin; the second, 28 parts of silver, to 2 of copper, to 3 of tin; and the third, 25 parts of silver, to 4 of copper, to 4 of tin. How many ounces of each of these alloys melted together will form 600 oz. of an alloy consisting of 8 parts of silver, to 1 of copper, to 1 of tin?

48. If Prob. 47 demanded merely that the alloy should contain 8 parts of silver to 1 of copper (without specifying the amount of tin), how many ounces of each of the given alloys would then be required? Why is this problem indeterminate (cf. § 107)?

107.* **Determinate and indeterminate systems of equations.** As we have already seen, a system containing as many independent equations as unknown numbers, can always be solved, *i.e.*, the unknown numbers can be determined (§§ 101–106). Such a system is, therefore, a **determinate system.**

On the other hand, a system in which there are fewer independent equations than unknown numbers is an **indeterminate system.** It is easy to show that this statement — already seen to be true in the case of a single equation containing two unknown numbers (§ 99) — is true generally.

Thus, suppose we have three equations containing four unknown numbers. By regarding one of these numbers temporarily as known, we can solve the given equations for the other three; *i.e.*, we can express any three of the four unknown numbers *in terms of the fourth.* To *every* assigned value, therefore, of this fourth unknown number, there corresponds a set of values of the other three (cf. § 99); hence the system is indeterminate.

Again, there can never be in a system more independent equations than there are unknown numbers.

For, if that were possible, suppose there are three independent equations, viz.,

$$ax + by = c, \tag{1}$$

$$hx + jy = k, \tag{2}$$

and

$$lx + my = n, \tag{3}$$

containing but two unknown numbers, x and y.

On solving (1) and (2) we obtain

$$x = \frac{cj - bk}{aj - bh} \text{ and } y = \frac{ch - ak}{bh - aj},$$

and on substituting these values for x and y in (3), we obtain

$$l\left(\frac{cj - bk}{aj - bh}\right) + m\left(\frac{ch - ak}{bh - aj}\right) = n;$$

* This article may be omitted till the subject is reviewed.

i.e., the known numbers of these equations are *not* independent (n, for example, is expressed in terms of a, b, c, l, etc.), hence the given equations are themselves not independent.

REVIEW EXERCISE—CHAPTERS VI–X

Find the H. C. F. and also the L. C. M. of:

1. $6x^2+13x-5$ and $3x^3+2x^2+2x-1$.

2. $12x^2-29x+14$ and $8x^2-30+6x^4+11x^3+33x$.

3. Change $\frac{5a-1}{6x(2+ax)}$ to an equal fraction whose denominator is $24x-6a^2x^3$; also to an equal fraction whose numerator is $1-10ay-5a+2y$.

Simplify:

4. $\dfrac{ax^m-bx^{m+1}}{a^2bx-b^3x^3}$.

5. $\dfrac{x^{q-1}+y^{p-1}}{x^{2(q-1)}-y^{2(p-1)}}$.

6. $\dfrac{r^4-13r^2+36}{r^4-r^3-7r^2+r+6}$.

7. $\dfrac{\dfrac{1+x}{1+x^2}-\dfrac{1+x^2}{1+x^3}}{\dfrac{1+x^2}{1+x^3}-\dfrac{1+x^3}{1+x^4}}$.

8. $\dfrac{1+\dfrac{4x^2}{6xy+9y^2}}{1+\dfrac{9y^2}{4x^2-6xy}}\div\dfrac{8x^3-27y^3}{81y^4}$.

9. $\dfrac{1}{(a-b)(b-c)}+\dfrac{1}{(c-b)(c-d)}-\dfrac{1}{(d-c)(b-a)}$.

10. $\dfrac{k-1}{(k-l)(k-m)(n-k)}-\dfrac{1-k}{(l-k)(n-k)(k-p)}$.

11. Why may a term be transposed from one member of an equation to the other by merely changing its sign?

12. When are equations conditional? identical? integral? fractional? literal? numerical? indeterminate? Illustrate each of your answers.

Solve, and check as the teacher directs:

13. $3x^2-5x-12=0$.

14. $6m^2-13m=-6$.

15. $\dfrac{2+x}{2-x}=-\dfrac{19}{21}$.

16. $1-\frac{2}{s}-\frac{3}{2s}=\frac{4}{3s}$.

17. $\dfrac{\frac{1}{x}+\frac{1}{a}}{\frac{3}{x}\cdot\frac{2}{a}}=a+b$.

18. $6x^4+7x^3-20x^2=0$.

19. $(x-2)^2+(x+5)^2=(x+7)^2$.

20. $7x+5\left(1-\frac{3x}{b}\right)=a(x-a)$.

21. $\frac{3}{2v+5}+\frac{4}{2v-5}=\frac{7-8v}{25-4v^2}$.

22. $\frac{y-4}{y-5}-\frac{y-6}{y-7}=\frac{y-5}{y-6}-\frac{y-7}{y-8}$.

23. $(x-a)(a-b+c)=(x+a)(b-a+c)$.

24. Show that while 2 is a root of the integral equation which results from clearing $\frac{3x}{x+5}+\frac{42}{(x+5)(x-2)}=8+\frac{6}{x-2}$ of fractions, it is not a root of the given fractional equation. How could we avoid introducing this extraneous root?

25. Form the equations whose roots are: $2, -9$; $-3\frac{1}{2}, 4$; $\frac{4}{5}, \frac{3}{7}$; $-2a, -6a$; $1, 3, -7$; $1-c, c-1$.

26. When are two equations equivalent? inconsistent? simultaneous? independent? Illustrate each of your answers.

27. Explain the term "elimination" as applied to simultaneous equations, and outline three methods of elimination.

Solve the following systems; check as the teacher directs:

28. $\begin{cases}\frac{7-2x}{5-3y}=\frac{3}{2},\\ y-x=4.\end{cases}$

29. $\begin{cases}2x-\frac{y-3}{5}=4,\\ 3y+\frac{x-2}{3}=9.\end{cases}$

30. $\begin{cases}\frac{1}{x}+\frac{1}{y}=\frac{1}{4},\\ \frac{1}{x}-\frac{1}{y}=\frac{1}{12}.\end{cases}$

31. $\begin{cases}5abx+2y=16b,\\ 3abx+4y=18b.\end{cases}$

32. $\begin{cases}\frac{x}{a}+\frac{y}{b}=1,\\ \frac{x}{3a}+\frac{y}{6b}=\frac{2}{3}.\end{cases}$

33. $\begin{cases}\frac{y-z}{2}-\frac{x+z}{4}=\frac{1}{2},\\ \frac{x-y}{5}-\frac{x-z}{6}=0,\\ \frac{y+z}{4}=\frac{x+y}{2}-4.\end{cases}$

34. If $ax^2 + bx + c$ becomes 8, 22, 42, respectively, when x becomes 2, 3, 4, what will it become when x becomes $-\frac{1}{3}$?

35. The sum of two numbers is 5760, and their difference is $\frac{1}{3}$ of the greater. Find the numbers.

36. What number added to its reciprocal gives 5.2?

37. It takes 2000 square tiles of a certain size to pave a hall, or 3125 square tiles whose dimensions are 1 inch less. Find the area of the hall floor. How many solutions has the equation of this problem? How many has the problem itself?

38. Divide the number a into two parts such that the second part shall equal n increased by m times the first part.

39. What number must be added to m and to n in order that the first sum divided by the second shall equal p/q? What does your answer become when $p = q$? What does this indicate (1) when $m = n$, (2) when m and n are unequal?

40. In order to build a new clubhouse, a country club assessed each of its 200 members a certain sum; later an increase of 50 in the membership reduced the individual assessments by \$10. Find the cost of the proposed house.

41. At what time between 3 and 4 o'clock is the minute-hand 25 minute spaces ahead of the hour-hand?

42. The freezing point of water is marked 0° on a Centigrade thermometer, and 32° above zero on a Fahrenheit thermometer. If 100° Centigrade = 180° Fahrenheit, find the reading on a Centigrade thermometer corresponding to 68° Fahrenheit. (Make a diagram of each scale.)

43. State and solve the *general* problem of which Prob. 42 is a particular case. By substitution in the formula thus obtained express in the Centigrade scale the following Fahrenheit readings: 44°; 212°; $-10°$; 0°.

44. A man rows a boat with the tide 8 miles in $1\frac{3}{5}$ hr. and returns against a tide $\frac{2}{5}$ as strong in 4 hr. What is the rate of the stronger tide? At what rate does the man row in still water?

45. A man selling eggs to a grocer counted them out of his basket 4 at a time and had 1 egg left over; the grocer counted them into his box 5 at a time and there were 3 left over. If the man had between 6 and 7 dozen eggs, how many must there have been (cf. § 99)?

46. Of two wheelmen, A and B, A starts c hours in advance of B, and travels at the rate of a miles in b hours, while B follows at the rate of p miles in q hours. How far will A travel before he is overtaken by B?

Under what conditions is this solution positive? negative? zero? infinite? *Interpret* the result in each case.

CHAPTER XI

INVOLUTION AND EVOLUTION

I. INVOLUTION

108. Introductory. For the meaning of the words *base*, *exponent*, and *power*, as used in algebra, see §§ 9, 30, and 36. The process of raising a number or expression to any given power is called **involution.**

In this chapter, as in the earlier treatment of powers, we shall use *only positive integers as exponents.* Later on (Chapter XVI), however, we shall find it advantageous to employ such symbols as a^0, a^{-5}, and $a^{\frac{2}{3}}$ also, and we shall then assign suitable meanings to such symbols.

109. Even powers, odd powers, powers of fractions, etc. A power of any given number is called **even** or **odd** according as its exponent is even or odd.

From the law of signs given in § 18 it follows that:

(1) All integral powers of a positive number are positive.

(2) All even integral powers of a negative number are positive.

(3) All odd integral powers of a negative number are negative.

And from § 83 (i) [cf. also Ex. 30, p. 121], it follows that

$$\left(\frac{a}{b}\right)^3=\frac{a^3}{b^3},\quad \left(\frac{m}{n}\right)^4=\frac{m^4}{n^4},\text{ etc.}$$

Let pupils fully explain each of the above statements.

EXERCISE LXXIV

1. Answer again questions 18–20 on p. 39.

2. Write that power whose base is k and whose exponent is $m-3$. Are there any limitations here on the value of k? on the value of m?

3. From the definition of an exponent show that $x^3 \cdot x^5 = x^8$. Also that $2^4 \cdot 2 \cdot 2^2 = 2^7$.

4. For what values of n between 1 and 10 is $(-3)^n \cdot (-5)^{2n}$ positive? Explain.

5. Show that an even power of a negative number is positive.

6. How is a fraction raised to a power (cf. Ex. 30, p. 121)? Illustrate your answer.

Simplify each of the following expressions:

7. $\left(\frac{-3a^2}{5x^4}\right)^3.$

8. $\left(-\frac{3a^2}{5x^4}\right)^3.$

9. $\left(\frac{2u^3}{x-1}\right)^2.$

10. $\left(-\frac{\frac{1}{2}}{5-3k^2}\right)^3.$

110. Exponent laws. The following formulas state what are known as the *exponent laws.* The bases (a, b, and c) stand for any numbers or algebraic expressions whatever, but the exponents are positive integers.

(i) *First exponent law.* $a^m \cdot a^n = a^{m+n}.$ [§ 30

For, just as
$$a^3 \cdot a^2 = (a \cdot a \cdot a) \cdot (a \cdot a)$$
$$= a^5, \text{ i.e., } a^{3+2};$$
so, too,
$$a^m \cdot a^n = (a \cdot a \cdot a \cdots \text{to } m \text{ factors})(a \cdot a \cdot a \cdots \text{to } n \text{ factors})$$
$$= a \cdot a \cdot a \cdots \text{to } (m+n) \text{ factors}$$
$$= a^{m+n}.$$

Similarly, $a^m \cdot a^n \cdot a^p = a^{m+n+p}.$

(ii) *Second exponent law.* $(a^m)^n = a^{mn}$.

For, just as $(a^3)^2 = (a \cdot a \cdot a)^2$
$= (a \cdot a \cdot a) \cdot (a \cdot a \cdot a)$
$= a^6$, *i.e.*, $a^{3 \cdot 2}$;

so, too, $(a^m)^n = (a \cdot a \cdot a \cdots \text{ to } m \text{ factors})^n$
$= a \cdot a \cdot a \cdots \text{ to } mn \text{ factors}$
$= a^{mn}$.

(iii) *Third exponent law.* $a^n \cdot b^n = (ab)^n$.

For, just as $a^3 \cdot b^3 = a \cdot a \cdot a \cdot b \cdot b \cdot b$
$= ab \cdot ab \cdot ab$
$= (ab)^3$;

so, too,

$$a^n b^n = (a \cdot a \cdot a \cdots \text{ to } n \text{ factors}) \cdot (b \cdot b \cdot b \cdots \text{ to } n \text{ factors})$$
$$= ab \cdot ab \cdot ab \cdots \text{ to } n \text{ factors}$$
$$= (ab)^n.$$

Similarly, $a^n b^n c^n = (abc)^n$.

(iv) *Fourth exponent law.* $a^m \div a^n = a^{m-n}$. [§ 36

This law is an immediate consequence of (i) above, and of the definition of division (§ 8), for since

$$a^{m-n} \cdot a^n = a^{m-n+n} = a^m,$$

therefore $a^m \div a^n = a^{m-n}$.

EXERCISE LXXV

Simplify, and explain your work in each case:

1. $a^5b^2 \cdot ab^3$.
2. $3x^4y \cdot (-2x^2y^6)$.
3. $a \cdot a^2 \cdot a^7 \cdot a^3$.
4. $\dfrac{x^5y^2}{x^2y}$.
5. $\dfrac{-c^3d^6e^2}{cd^2e}$.
6. $(x^2z)^3$.
7. $(-5a)^2$.
8. $\left(\dfrac{-ax}{y^2}\right)^7$.
9. $(\frac{2}{3}r^5s^3)^4$.
10. $\left(\dfrac{-2mn}{p^2}\right)^5$.
11. $\left(\dfrac{s^3}{7st^2}\right)^2$.
12. $x^m \cdot x^3$.
13. $x^m \div x^3$.
14. $(x^m)^3$.
15. $(2x^{2m})^5$.
16. $s^a \cdot s^{3a}$.
17. $v^m \cdot v^n \cdot v^2$.
18. $\left(\dfrac{3x^a}{y^b}\right)^2$.

19. $\dfrac{a^{2m}b^{3n}}{a^3b^n}$.

20. $\left(\dfrac{r^{a-1}}{r}\right)^3$.

21. $(k^2l^3)^b$.

22. $(-c)^{2m}$.

23. $\left(-\dfrac{c}{d^2}\right)^{2m+1}$.

24. $\left(\dfrac{u^m}{v^n}\right)^c$.

25. $(u^a v^{2b-1})^c$.

26. $(5\,r^m)^n$.

27. $\left(\dfrac{x+1}{x+2}\right)^2$.

28. $\left(\dfrac{c-2}{c^5}\right)^3$.

29. $\left[\left(\dfrac{a-3\,b}{a+3\,b}\right)^2\right]^2$.

Write the following as powers of products [cf. law (iii) above]:

30. h^2k^2.

31. $r^5s^5t^5$.

32. c^2d^4.

33. x^3y^6.

34. x^8y^4.

35. $-2^3 \cdot 3^3$.

36. $a^x \cdot (2\,b)^x$.

37. $3^4 \cdot (-m)^4 \cdot (n)^4$.

38. $x^{2a} \cdot y^{2a} \cdot z^{4a}$.

39. What does a represent in the proofs of § 110? May it represent a polynomial as well as a number?

40. Translate the first exponent law (§ 110) into verbal language (cf. § 30).

41. Translate the second, third, and fourth exponent laws into verbal language.

42. Is $(a \cdot b \cdot c)^2$ equal to $a^2 \cdot b^2 \cdot c^2$? Is $(a+b+c+d)^2$ equal to $a^2+b^2+c^2+d^2$? Explain your answers.

43. Is $[(-2)^3]^2$ equal to $[(-2)^2]^3$? Why? Is $(x^3)^4$ equal to $(x^4)^3$? Why?

111. Powers of binomials. We have already seen (§§ 52 and 57) that

$$(a+b)^2 = a^2 + 2\,ab + b^2,$$

and

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

These powers (expansions) were obtained by direct multiplication, and the higher powers may, of course, be obtained in the same way. Thus,

$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4,$$

$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5,$$

$$(a+b)^6 = a^6 + 6\,a^5b + 15\,a^4b^2 + 20\,a^3b^3 + 15\,a^2b^4 + 6\,ab^5 + b^6, \text{ etc.}$$

The following questions may serve to bring out the striking similarity of these expansions:

1. How does the exponent of the first term in each expansion compare with that of the corresponding binomial?

2. How, in each expansion, does the exponent of a change as we pass from term to term toward the right?

3. In which term of each expansion does b first appear? How does the exponent of b change from term to term?

4. How many terms in each expansion? What is the sign of each term?

5. What coefficient has the first term of each expansion? the second term?

6. Multiply the coefficient of any term in any of the expansions by the exponent of a in that term, and divide this product by the number of the term; how does this quotient compare with the coefficient of the next term?

7. Assuming that the expansion of $(a+b)^8$ is similar in form to the expansion of $(a+b)^2$, $(a+b)^3$, etc., complete the statement:

$$(a+b)^8 = a^8 + 8\,a^7b + 28\,a^6b^2 + \cdots.$$

112. Binomial theorem. (i) The answers to the first six questions in § 111, when combined, may be expressed symbolically thus:

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots.$$

This formula, which was discovered by the celebrated English mathematician Sir Isaac Newton (1642–1727), is called the **binomial theorem**; its correctness is proved in §§ 206–207.

(ii) Since $a - b = a + (-b)$,

therefore

$$(a-b)^3 = [a+(-b)]^3 = a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3$$
$$= a^3 - 3a^2b + 3ab^2 - b^3,$$

i.e., $(a-b)^3$ differs from $(a+b)^3$ only in having the signs of its even terms negative. So also for other powers of $a-b$, *i.e.*,

$$(a-b)^n = a^n - \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots.$$

EXERCISE LXXVI

Write down the expansions of the following binomials:

1. $(a+x)^4$. **5.** $(a+c)^8$. **9.** $(m^2+b)^3$.

2. $(m+t)^3$. **6.** $(x+y)^4$. **10.** $(m+b^2)^5$.

3. $(u+v)^6$. **7.** $(y+z)^6$. **11.** $(m^2+b^2)^7$.

4. $(p+q)^5$. **8.** $(k+l)^7$. **12.** $(m^2+b^3)^6$.

13. Expand each of the following expressions: $(x+y)^2$, $(x+y)^3$, and $(x+y)^5$; then multiply the first two expanded forms together and thus verify that $(x+y)^2 \cdot (x+y)^3 = (x+y)^5$.

14. What terms in the expansion of $(c-d)^5$ are negative? Why?

15. Write the first five terms of $(s-2t)^8$ and simplify your result (cf. Ex. 3, p. 71).

16. How many terms are there in the expansion of $(m+n)^5$? How many in $(a-b)^8$? How many in $(3s-2t)^n$?

Write each of the following expressions in its expanded form:

17. $(k-c)^4$. **23.** $(4c+d^5)^3$. **29.** $(v^3-2)^3$.

18. $(r-s)^8$. **24.** $(mn-rs)^5$. **30.** $(2xy-1)^6$.

19. $(m-n)^6$. **25.** $(ab+cd^2)^7$. **31.** $(c+a)^9$.

20. $(c+d)^{10}$. **26.** $(3a^2+c^3d)^4$. **32.** $(2m+3)^4$.

21. $(x^2+y)^5$. **27.** $(k+1)^7$. **33.** $(2-3c^2d)^5$.

22. $(2r-a^2)^4$. **28.** $(u-2)^5$. **34.** $(2-a^2b^2)^7$.

35. $(s^2 - sv^3)^8$.

36. $(2xy^2 + x^2y)^5$.

37. $\left(x + \frac{2}{x}\right)^4$.

38. $\left(r - \frac{r^2}{3}\right)^5$.

39. $\left(\frac{2}{s} + \frac{3}{t}\right)^3$.

40. $[a + (b + c)]^3$.

41. $(c + d + e)^3$.

42. $(c - d + e)^3$.

43. $(2x^2 - m - 1)^4$.

44. Write the first four terms of $(a + x)^{25}$; the first three terms of $(x - y)^{32}$; the first three terms of $(2ax - 3k^2)^9$.

II. EVOLUTION

113. Definitions. Here, as in arithmetic, by the **square root** of any given number we mean a number whose square equals the given number.

Thus, since $7^2 = 49$, therefore 7 is a square root of 49.

Similarly, the third or **cube root** of a number is a number whose third power equals the given number.

Roots are usually indicated by the radical sign ($\sqrt{\ }$), which is a modification of the letter *r*, the initial letter of the Latin word *radix*, meaning root. A small figure, called the **index** of the root, is written in the opening of the radical sign to indicate the particular root to be extracted. When no index is written, the index is understood to be 2.

E.g., $\sqrt{a}$ indicates the second or square root of a, (1)

$\sqrt[3]{a}$ indicates the third or cube root of a, (2)

and $\sqrt[7]{a}$ indicates the seventh root of a, (3)

and, in general, $\sqrt[n]{a}$ means the nth root of a,

i.e., $$(\sqrt[n]{a})^n = a. \qquad (4)$$

An indicated root is said to be an **even root** or an **odd root** according as its index is an even or an odd number.

The process of finding a root of any given number is called **evolution**; it is the inverse of involution (cf. § 108).

NOTE. In practice the radical sign is usually combined with a vinculum (§ 11) to indicate clearly just how much of the expression following the radical sign is to be affected by that sign; thus $\sqrt{9 + 16}$ means the square root of the sum of 9 and 16, while $\sqrt{9} + 16$ indicates that 16 is to be added to the square root of 9.

114. Law of signs of roots. From the definition of root (§ 113), and from § 109, it follows that:

1. An odd root of any number has the same sign as the number itself. Thus, $\sqrt[3]{8}=2$, and $\sqrt[3]{-8}=-2$, because $2^3=8$ and $(-2)^3=-8$.

2. An even root of a positive number has two opposite values, *i.e.*, one positive, the other negative. Thus, $\sqrt[4]{81}=+3$ or -3, since $(+3)^4=(-3)^4=81$. Instead of writing $\sqrt[4]{81}=+3$ or -3, we usually write $\sqrt[4]{81}=\pm 3$; this expression is read, "The fourth root of 81 equals plus or minus 3."

3. An even root of a negative number is neither a positive nor a negative number. Thus, $\sqrt{-9}$ is neither $+3$ nor -3, since $(+3)^2=(-3)^2=+9$, and not -9.

NOTE. Such indicated roots as $\sqrt{-9}$ are called **imaginary numbers** (cf. §§ 146, 164); all other numbers are, for distinction, called **real numbers.** To provide for such roots as $\sqrt{-9}$ we must again extend the number system, just as we did when subtractions like $3-8$ first presented themselves (cf. Chap. II).

115. Roots of monomials. If a monomial is an exact power, the corresponding root can usually be written down by inspection.

E.g., $\sqrt[3]{8a^6x^3}=2a^2x$, because $(2a^2x)^3=8a^6x^3$ (§ 110);

$\sqrt{9x^4y^6}=\pm 3x^2y^3$, because $(+3x^2y^3)^2=(-3x^2y^3)^2=9x^4y^6$;

$\sqrt[5]{-32x^{10}}=-2x^2$, because $(-2x^2)^5=-32x^{10}$;

$$\sqrt[3]{\frac{8m^3}{x^3y^6}}=\frac{2m}{xy^2},\text{ because }\left(\frac{2m}{xy^2}\right)^3=\frac{8m^3}{x^3y^6}.$$

EXERCISE LXXVII

1. What is meant by the square root of a number? Are both 5 and -5 square roots of 25? Why?

2. What are the square roots of 64? the fourth roots of 16? Why? If a is any *even* root of a number, then $-a$ also is a root (with the same index) of that number, — explain.

3. What is the cube root of 27? of -27? of 64? of -64? Explain. How does $\sqrt[5]{32}$ compare with $\sqrt[5]{-32}$?

4. How does the *sign* of an odd root of a number compare with the sign of the number itself? Why? Answer these questions for an even root also.

5. Give the cube of each integer between 1 and 7. Name the cube root of: -8; 1000; -1728; $-\frac{27}{64}$; $\frac{125}{343}$; -216; 8000.

6. What is the sign of any *even* power of a positive or negative number? Can, then, an even root of a negative number be positive? negative? Illustrate your answer.

7. Is -13 a square root of 169? Why? Is $5\,as^3$ the cube root of $125\,a^3s^9$? Why? How can you tell whether one given number is a square root of another given number? a fifth root?

8. How do we find the exponents in the cube root of $8\,a^{12}x^3y^6$? in the 4th root of $a^8b^{12}c^4$? in the 6th root of $m^{12}n^{30}$? in the nth root of $a^{3n}b^{2n}x^{kn}$? Explain.

Find the following indicated roots, and check your answers. Also, tell which are even and which are odd roots, and name the index in each case:

9. $\sqrt[3]{a^3b^6c^{15}}$.

10. $\sqrt{16\,a^4x^6y^2}$.

11. $\sqrt[5]{32\,x^5y^{10}}$.

12. $\sqrt[5]{-243\,a^{10}x^5}$.

13. $\sqrt[n]{a^{4n}x^{3n}y^{2n}}$.

14. $\sqrt[3]{-64\dfrac{x^6}{y^9}}$.

15. $\sqrt[5]{\dfrac{-32\,a^5x^{40}}{243\,y^{25}}}$.

16. $\sqrt[7]{128\,a^{7x}b^{14y}}$.

17. $\sqrt[3]{-\dfrac{125\,x^{12}y^6}{1728\,a^6z^9}}$.

18. $\sqrt[7]{\dfrac{-(x-y)^{14}}{128\,x^{14}}}$.

19. $\sqrt{\dfrac{169}{400}\,s^{18}t^{10}}$.

20. $\sqrt[6]{\dfrac{64\,m^6p^{12}}{729\,h^{24}z^6}}$.

21. $\sqrt[3]{\dfrac{.027\,a^3x^6}{.064\,b^3z^6}}$.

22. $\sqrt[3]{\dfrac{216\,c^3d^{18}}{343\,(c-d)^3}}$.

23. $\sqrt{\dfrac{121}{225}\,m^4b^{8n}}$.

24. $\sqrt[3]{\dfrac{1000\,c^{3a}}{125\,d^{15}}}$.

25. $\sqrt[2n]{\dfrac{a^{2kn-2n}b^{4n}}{x^{6n}y^{4n}}}$.

26. $\sqrt[x]{\dfrac{a^{5x}b^{xz}x^{3x}}{2^{2x}y^{4x}z^x}}$.

27. $\sqrt[4]{\dfrac{81\,(a-c)^{12m}}{625\,a^{4m-8}}}$.

28. Write a rule for the extraction of such roots as the above, emphasizing particularly the matter of exponents and signs. Does your rule apply to roots of polynomials also?

29. Is $\sqrt{9 \cdot 16}$ equal to $\sqrt{9} \cdot \sqrt{16}$? Why? Is $\sqrt{9+16}$ equal to $\sqrt{9}+\sqrt{16}$? Is $\sqrt{a^2+b^2}$ equal to $\sqrt{a^2}+\sqrt{b^2}$ (cf. Ex. 42, p. 173)? Is $\sqrt{a^2 \cdot b^2}$ equal to $\sqrt{a^2} \cdot \sqrt{b^2}$? State in words your conclusion as to the square roots of sums and products.

116. Roots of polynomials extracted by inspection. If a polynomial is an exact power of a *binomial*, or the square of a polynomial, a little study usually reveals the corresponding root.

Ex. 1. Find the square root of $m^6 + 4\,m^3n + 4\,n^2$.

SOLUTION. This expression is easily seen to be $(m^3 + 2\,n)^2$; therefore $\sqrt{m^6 + 4\,m^3n + 4\,n^2} = \pm (m^3 + 2\,n)$.

Ex. 2. Find the cube root of $8\,a^3 - 36\,a^2b - 27\,b^3 + 54\,ab^2$.

SOLUTION. This polynomial consisting of four terms, two of which, viz., $8\,a^3$ and $-27\,b^3$, are exact cubes, *may* be the cube of a binomial (§ 57); if so, that binomial must be $2\,a - 3\,b$. (Why?)

On cubing $2\,a - 3\,b$, we see that

$$\sqrt[3]{8\,a^3 - 36\,a^2b - 27\,b^3 + 54\,ab^2} = 2\,a - 3\,b.$$

Ex. 3. Find the square root of $a^2 + b^2 - 2\,ab - 4\,bc + 4\,c^2 + 4\,ac$.

SOLUTION. This polynomial consisting of six terms, three of which are exact squares, and three of which are double products, *may* be the square of a trinomial whose terms are the square roots of the square terms (§ 56). A little further examination shows that $\sqrt{a^2 + b^2 - 2\,ab - 4\,bc + 4\,c^2 + 4\,ac} = \pm (a - b + 2\,c)$.

EXERCISE LXXVIII

By inspection find the following roots, and check results:

4. $\sqrt{4\,x^2 + 12\,x + 9}$.

5. $\sqrt{25\,y^2 - 40\,y + 16}$.

6. $\sqrt{(m+n)^2 - 4\,(m+n) + 4}$.

7. $\sqrt{x^2 + 2\,xy + y^2 - 2\,xz - 2\,yz + z^2}$.

8. $\sqrt[3]{8\,u^3 - 12\,u^2v - v^3 + 6\,uv^2}$.

9. $\sqrt[4]{x^4 - 4\,x^3y + y^4 - 4\,xy^3 + 6\,x^2y^2}$.

10. $\sqrt[3]{8\,h^3 - 84\,h^2k + 294\,hk^2 - 343\,k^3}$.

11. $\sqrt[5]{a^5 - b^5 - 5\,a^4b + 5\,ab^4 + 10\,a^3b^2 - 10\,a^2b^3}$.

12. $\sqrt{a^2 + 9\,b^2 - 6\,ab + 6\,(x - 2\,y)(a - 3\,b) + 9\,(x^2 - 4\,xy + 4\,y^2)}$.

13. $\sqrt[6]{x^6 - 6\,abx^5 + 15\,a^2b^2x^4 - 20\,a^3b^3x^3 + 15\,a^4b^4x^2 - 6\,a^5b^5x + a^6b^6}$.

117. Square roots of polynomials.*

Since, $$(k + u)^2 = k^2 + 2\,ku + u^2,$$

therefore $$\sqrt{k^2 + 2\,ku + u^2} = k + u,$$

and we shall now try to find a method by which the root, $k + u$, may be found from the power, $k^2 + 2\,ku + u^2$.

Manifestly the *first term* of the root (viz., k) is the square root of the first term of the power.

And having subtracted k^2 (the square of this root term) from the power, the *next term* of the root (viz., u) is found by dividing the first term of the remainder (viz., $2\,ku + u^2$) by twice the root term previously found (viz., $2\,k$).

The actual work may be arranged thus:

$$\begin{array}{lll} & k^2 + 2\,ku + u^2 & \underline{\mid k + u} \\ & \underline{k^2} & \\ \text{Trial divisor}^{\dagger} = 2\,k & \mid 2\,ku + u^2 & \\ \text{Complete divisor} = 2\,k + u & \mid \underline{2\,ku + u^2} = (2\,k + u)\,u & \\ & \quad 0 & \end{array}$$

The same method may be applied to other polynomials.‡

Ex. 1. Find the square root of $4\,s^2 - 28\,st^3 + 49\,t^6$.

SOLUTION. Let k represent the first term of this root, and u the next term; then

$4\,s^2 - 28\,st^3 + 49\,t^6$ *contains* $(k + u)^2$, *i.e.*, $k^2 + 2\,ku + u^2$.

Now the first term of the square root of $4\,s^2 - 28\,st^3 + 49\,t^6$ is, manifestly, $2\,s$, *i.e.*, $k = 2\,s$; and the next root term may be found as above, thus:

---

* For a more detailed discussion of this topic, see *El. Alg.* § 125.

† Twice the root already found at any stage of the work is usually called the trial divisor (T. D.) and the trial divisor plus the next root term is called the complete divisor (C. D.). ‡ See Note 1, p. 181.

$$
\begin{array}{ll|l}
 & 4\,s^2 - 28\,st^3 + 49\,t^6 & \underline{2\,s - 7\,t^3} \\
k^2 = (2\,s)^2 = & \underline{4\,s^2} & \\
\text{T. D.} = 2\,k = 4\,s & \quad -28\,st^3 + 49\,t^6 & \\
\text{C. D.} = 2\,k + u = 4\,s - 7\,t^3 & \quad \underline{-28\,st^3 + 49\,t^6} = (2\,k + u)\,u & \\
 & \qquad\qquad 0 &
\end{array}
$$

CHECKS: (1) Square $2\,s - 7\,t^3$, or (2) substitute special values for s and t (cf. § 25).

Ex. 2. Find the square root of $9\,x^4 + 6\,x^3 - 11\,x^2 - 4\,x + 4$.

SOLUTION. At any stage of the process of finding this root, let k represent the term or terms already *known*, and let u represent the *next* term; then

$9\,x^4 + 6\,x^3 - 11\,x^2 - 4\,x + 4$ *contains* $k^2 + 2\,ku + u^2$. [§ 56

Here the first term of the root is $3\,x^2$, *i.e.*, $k = 3\,x^2$, and the next term (u) may be found as in Ex. 1, thus:

$$
\begin{array}{ll|l}
 & 9\,x^4 + 6\,x^3 - 11\,x^2 - 4\,x + 4 & \underline{3\,x^2 + x - 2} \\
k^2 = (3\,x^2)^2 = & \underline{9\,x^4} & \\
\text{T. D.} = 2\,k = 6\,x^2 & \quad 6\,x^3 - 11\,x^2 - 4\,x + 4 & \\
\text{C. D.} = 2\,k + u = 6\,x^2 + x & \quad \underline{6\,x^3 + x^2} \qquad = (2\,k + u)\,u & \\
\text{T. D.} = 2\,k^* = 6\,x^2 + 2\,x & \qquad -12\,x^2 - 4\,x + 4 & \\
\text{C. D.} = 2\,k + u = 6\,x^2 + 2\,x - 2 & \qquad \underline{-12\,x^2 - 4\,x + 4} = (2\,k^* + u)\,u & \\
 & \qquad\qquad\qquad 0 &
\end{array}
$$

CHECKS: (1) Square $3\,x^2 + x - 2$; or (2) use § 25.

NOTE 1. Before applying the process of Exs. 1 and 2 a polynomial should be arranged according to ascending or descending powers of one of its letters.

NOTE 2. Exs. 1 and 2 show how to find the square root of a polynomial *which is an exact square; i.e.*, if the above process is continued until a zero remainder is reached, then the square of the root thus found will be the given polynomial. If, however, the same process is applied to a polynomial which is not an exact square, then as many root terms as desired may be found, and the square of this root, at any stage of the work, equals the result of subtracting the corresponding remainder from the given polynomial; such a root is called an **approximate** root, and also the **root to n terms.**

---

* Here k represents $3\,x^2 + x$, and u represents -2. Observe also that the first and second subtractions in this solution are together equivalent to the subtraction of $(3\,x^2 + x)^2$ from the given expression.

EXERCISE LXXIX

Find the square root of each of the following expressions, and check your work:

3. $9m^4 - 66m^2 + 121.$

4. $16r^6 + 104r^3 + 169.$

5. $4 + 8x - 4x^3 + x^4.$

6. $1 + 2m - 3m^2 - 4m^3 + 4m^4.$

7. $1 - 6y + 5y^2 + 12y^3 + 4y^4.$

8. $9a^4 + 30a^3x + a^2x^2 - 40ax^3 + 16x^4.$

9. $4x^6 + 17x^2 - 22x^3 + 13x^4 - 24x - 4x^5 + 16.$

10. $4a^4 + 64b^4 - 20a^3b + 57a^2b^2 - 80ab^3.$

11. $6x^5y + 2x^3y^3 - 28xy^5 + 9x^6 + 4y^6 + 45x^2y^4 + 43x^4y^2.$

12. $3x^4 - 2x^5 - x^2 + 2x + 1 + x^6.$

13. $48a^4 + 12a^2 + 1 - 4a - 32a^3 + 64a^6 - 64a^5.$

14. $46x^2 + 25x^4 - 44x^3 - 40x + 4x^6 + 25 - 12x^5.$

15. $x^4 - 2x^2y + 2x^2z^2 - 2yz^2 + y^2 + z^4.$

16. $\dfrac{x^4}{a^2} + 16a^2y^6 + 8x^2y^3.$

17. $x^2 + 2x - 1 - \dfrac{2}{x} + \dfrac{1}{x^2}.$*

18. $9x^2 - 24x + 28 - \dfrac{16}{x} + \dfrac{4}{x^2}.$

19. $4a^2 - 20a + 21 + \dfrac{10}{a} + \dfrac{1}{a^2}.$

20. $n^4 + 4n^3 + \dfrac{1}{n^2} + 2n + 4 + 4n^2.$

21. $x^4 + \dfrac{1}{x^4} + 4x^3 + \dfrac{1}{x^3} + 6x^2 + \dfrac{9}{4x^2} + 5 + 5x + \dfrac{5}{x}.$

22. $\dfrac{9e^2}{f^2} - \dfrac{6e}{f} + 4 - \dfrac{f}{e} + \dfrac{f^2}{4e^2}.$

23. $(x-y)^2 - 2(xy + xz - y^2 - yz) + (y+z)^2.$

24. $x^{2r} - 6x^{r+1}y - 30x^ry^2 + 10x^{2r-1}y + 25x^{2r-2}y^2 + 9x^2y^2.$

25. $1 + x$, to three terms (cf. Note 2, p. 181).

---

* Observe that this expression is already arranged according to descending powers of x.

26. $1 + 2m^2$, to four terms.

27. $a^2 + 1$, to three terms.

28. $1 + x - x^2$, to four terms.

29. $x^4 + 2x^3y + y^4 + xy^3 + x^2y^2$, to four terms.

30. In Ex. 1 is not $-2s$, as well as $+2s$, a square root of the first term? Solve Ex. 1, using $-2s$. Does your result check?

31. Solve Ex. 2, using $-3x^2$ as the square root of the first term, and compare your answer with that found in the text.

32. By extracting the square root until a numerical remainder is reached, show that $x^4+4x^3+8x^2+8x-5$ equals $(x^2+2x+2)^2-9$, and thus find the factors of $x^4+4x^3+8x^2+8x-5$.

33. As in Ex. 31, find the factors of $x^4+6x^3+11x^2+6x-8$; also of $a^6-6a^4+10a^3+9a^2-30a+9$.

118. Square roots of arithmetical numbers.* In order to proceed systematically, and find the successive digits of the root in their order from left to right, we first separate the given number into *periods* of two figures each, toward the right and left from the decimal point. The root may then be extracted by virtually the same process as that used in § 117.

Note. The reason for the separation into periods lies in this: the square of any number of tens ends in two ciphers, and hence the first two digits at the left of the decimal point are useless when finding the tens' digit of the root; they are, therefore, set aside until needed to find the units' digit of the root. So, too, the square of any number of hundreds ends in four ciphers, and hence, for a like reason, two periods are set aside when the hundreds' digit of the root is being found, and so on. Similarly for the periods at the right of the decimal point.

Ex. 1. Find the square root of 1156.

Solution. This number consists of two periods, hence its square root consists of two digits. Again, since 9 is the greatest

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* For a more complete discussion of this topic see *El. Alg.* § 126.

square in the left-hand period, therefore 3 is the first figure in the root. Now, let k represent the known part of the root at any stage of the work, and u the next root figure, then

$$1156 \textit{ contains } k^2 + 2\,ku + u^2,$$

and the work may be arranged as follows:

$$\begin{array}{llr|l}
 & & & k + u \\
 & & 11'56 & 30 + 4 = 34 \\
 & k^2 = (30)^2 = & 900 & \\
\text{T. D.} = 2\,k = 60 & & 256 & \\
\text{C. D.} = 2\,k + u = 60 + 4 & & 256 & = (2\,k + u)\,u \\
 & & 0 &
\end{array}$$

Check: $(34)^2 = 1156$.

Ex. 2. Find the square root of 315844.

Solution. Using k and u as in Ex. 1, we have

$$315844 \textit{ contains } k^2 + 2\,ku + u^2,$$

and the work may be arranged as follows:

$$\begin{array}{lr|l}
 & 31'58'44 & 500 + 60 + 2 = 562 \\
k^2 = (500)^2 = & 250000 & \\
\text{T. D.} = 2\,k = \quad 1000 & 65844 & \\
\text{C. D.} = 2\,k + u = 1060 & 63600 & = (2\,k + u)u \\
\text{T. D.} = 2\,k^* \quad = 1120 & 2244 & \\
\text{C. D.} = 2\,k + u = 1122 & 2244 & = (2\,k + u)u \\
 & 0 &
\end{array}$$

Check: $(562)^2 = 315844$.

Note. When some familiarity with the above process has been gained, the work may be abridged by omitting unnecessary ciphers, as shown below in finding the square root of 315844 and of 10.5625.

$$\begin{array}{r|l|l}
 & 31'58'44 & 562 \\
 & 25 & \\
106 & 658 & \\
 & 636 & \\
1122 & 2244 & \\
 & 2244 & \\
 & 0 &
\end{array}
\qquad\qquad
\begin{array}{r|l|l}
 & 10.'56'25 & 3.25 \\
 & 9 & \\
62 & 156 & \\
 & 124 & \\
645 & 3225 & \\
 & 3225 & \\
 & 0 &
\end{array}$$

---

* Here $k = 560$; compare footnote, p. 181.

EXERCISE LXXX

Extract the square root of each of the following numbers, and check your results:

3. 1296.	**6.** 9216.	**9.** 667489.	**12.** 17424.
4. 841.	**7.** 12.96.	**10.** 26.2144.	**13.** 36.8449.
5. 2209.	**8.** 62.41.	**11.** 1664.64.	**14.** 101.0025.

15. How may the square root of a fraction be found? Illustrate, using the fractions $\frac{9}{64}$ and $\frac{625}{729}$. Is $-\frac{25}{27}$ also a square root of the latter fraction? Why?

16. A number contains one decimal place; how many decimal places in its square? How many, if the number contains two decimal places? if it contains three? if it contains n? Explain.

17. Show from Ex. 16 that if the right-hand period of a *decimal* is incomplete, we must annex a cipher to complete it. Is this true of the left-hand period of an integral number also?

18. Extract the square root of 2 to two decimal places (cf. p. 181, Note 2). How many periods of ciphers must be annexed to 2 for this purpose? Why?

Find the square root of each of the following numbers, correct to two decimal places:

19. 13.5.	**21.** .017.	**23.** $\frac{5}{8}$.	**25.** $4\frac{2}{3}$.
20. $\frac{7}{9}$.	**22.** 1.1105.	**24.** $\frac{3}{11}$.	**26.** .049.

27. Is $\sqrt{36}$ equal to $\sqrt{9}\cdot\sqrt{4}$? Is $\sqrt{27}$ (*i.e.*, $\sqrt{9\cdot 3}$) equal to $3\sqrt{3}$ where the roots are extracted to two decimal places? to three decimal places?

28. Is $\sqrt{450}$ (*i.e.*, $\sqrt{225\cdot 2}$) equal to $15\sqrt{2}$ where the roots are correct to two decimal places? Show that $\sqrt{96}$ and $4\sqrt{6}$ are equal, to at least two decimal places.

119.* **Cube root of polynomials.** The procedure here is like that in § 117.

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* Articles 119, 120, with Exercises LXXXI and LXXXII, may, if the teacher prefers, be omitted till the subject is reviewed.

Since $\qquad (k+u)^3 = k^3 + 3k^2u + 3ku^2 + u^3,$

therefore $\qquad \sqrt[3]{k^3 + 3k^2u + 3ku^2 + u^3} = k + u.$

And this equation shows:

(1) that the *first term* of the cube root (viz., k) is the cube root of the first term of the polynomial;

(2) that the trial divisor for finding the *next term* of the root is $3k^2$;

(3) that the complete divisor is $3k^2 + 3ku + u^2$.

The actual work may be arranged thus:

$$\begin{array}{ll|l}
 & k^3 + 3k^2u + 3ku^2 + u^3 & \underline{k+u} \\
 & \underline{k^3\qquad\qquad\qquad\qquad} & \\
\text{T. D.} = 3k^2 & 3k^2u + 3ku^2 + u^3 & \\
\text{C. D.} = 3k^2 + 3ku + u^2 & \underline{3k^2u + 3ku^2 + u^3} = (3k^2 + 3ku + u^2)u & \\
 & 0 &
\end{array}$$

If now we let k represent the part of the root already known at any stage of the work, and let u represent the next term, then the above method will serve to extract the cube root of any "arranged" polynomial (cf. § 117, Exs. 1 and 2).

Thus, the cube root of $x^6 - 9x^5 + 30x^4 - 45x^3 + 30x^2 - 9x + 1$ may be found as follows:

$$\begin{array}{ll}
 & \underline{|\, x^2 - 3x + 1} \\
 & x^6 - 9x^5 + 30x^4 - 45x^3 + 30x^2 - 9x + 1 \\
 & (x^2)^3 = \underline{x^6 \qquad\qquad\qquad\qquad\qquad\qquad} \\
\text{T. D.} = 3(x^2)^2 = 3x^4 & -9x^5 + 30x^4 - 45x^3 + 30x^2 - 9x + 1 \\
\text{C. D.} = 3x^4 - 9x^3 + 9x^2 & \underline{-9x^5 + 27x^4 - 27x^3} \\
\text{T. D.} = 3(x^2 - 3x)^2 = 3x^4 - 18x^3 + 27x^2 & 3x^4 - 18x^3 + 30x^2 - 9x + 1 \\
\text{C. D.} = 3(x^2 - 3x)^2 + 3(x^2 - 3x) + 1 & \\
\quad = 3x^4 - 18x^3 + 30x^2 - 9x + 1 & \underline{3x^4 - 18x^3 + 30x^2 - 9x + 1} \\
 & 0
\end{array}$$

EXERCISE LXXXI

Find the cube root in Exs. 1–14, and check your results:

1. $8x^3 - 12x^2 + 6x - 1.$
2. $27x^3 - 189x^2y + 441xy^2 - 343y^3.$
3. $125n^3 - 150mn^2 - 8m^3 + 60m^2n.$
4. $225u^2v + 135uv^2 + 125u^3 + 27v^3.$

5. $x^6 - 20\,x^3 - 6\,x + 15\,x^4 - 6\,x^5 + 15\,x^2 + 1.$

6. $3\,x^5 + 9\,x^4 + x^6 + 8 + 12\,x + 13\,x^3 + 18\,x^2.$

7. $342\,x^2 - 108\,x - 109\,x^3 + 216 + 171\,x^4 - 27\,x^5 + 27\,x^6.$

8. $156\,x^4 - 144\,x^5 - 99\,x^3 + 64\,x^6 + 39\,x^2 - 9\,x + 1.$

9. $54\,x + \frac{8}{x^3} - 112 - \frac{48}{x^2} + \frac{108}{x} + x^3 - 12\,x^2$ (cf. Ex. 17, p. 182).

10. $20 + \frac{15}{c^2} + 15\,c^2 + c^6 + \frac{6}{c^4} + \frac{1}{c^6} + 6\,c^4.$

11. $\frac{30}{y} + \frac{8}{y^3} + 8\,y^3 + 30\,y - 12\,y^2 - 25 - \frac{12}{y^2}.$

12. $6\,a^5x^4 - 4\,a^3x^6 - 2\,a^6x^3 + 6\,a^2x^7 + 3\,a^8x + a^9 + x^9 - 3\,ax^8.$

13. $108\,y^5z - 27\,y^6 - 90\,y^4z^2 + 8\,z^6 - 80\,y^3z^3 + 60\,y^2z^4 + 48\,yz^5.$

14. $v^{3n} + 9\,v^{3n-3} + 21\,v^{3n-2} - 42\,v^{3n-4} - 36\,v^{3n-5} - 9\,v^{3n-1} - 8\,v^{3n-6}.$

15. Find the first three terms of $\sqrt[3]{1 + x}.$

16. Find the first four terms of $\sqrt[3]{1 - 3\,x + x^2}.$

120.* Cube root of numbers. The cube root of a number may be found by virtually the same process as that used in § 119 for finding the cube root of a polynomial (cf. §§ 118 and 117). The number should be separated into periods of 3 figures each, beginning at the decimal point (why ?), and the right-hand period, if incomplete, should be completed by annexing ciphers.

Ex. 1. Find the cube root of 42875.

SOLUTION

$$
\begin{array}{r|l|l}
 & & \;k + u \\
 & 42'875 & \underline{30 + 5 = 35} \\
k^3 = (30)^3 = & 27000 & \\ \hline
\text{T.D.} = 3\,k^2 = 3 \cdot (30)^2 = 2700 & 15875 & \\
\text{C.D.} = 3\,k^2 + 3\,ku + u^2 \qquad\qquad & & \\
= 3\,(30)^2 + 3\,(30)5 + 5^2 = 3175 & 15875 & \\ \hline
 & 0 &
\end{array}
$$

---

* See footnote, p. 185.

Ex 2. Find the cube root of 9825.17, correct to tenths.

SOLUTION

$$\begin{array}{r|l}
 & 9'825.'170 \quad \underline{|20 + 1 + .4} = 21.4 \\
k^3 = (20)^3 = & \underline{8000} \\
\text{T.D.} = 3\,k^2 = 3\,(20)^2 = 1200 \quad & 1825 \\
\text{C.D.} = 3\,k^2 + 3\,ku + u^2 \quad & \\
= 3\,(20)^2 + 3\,(20)\cdot 1 + 1^2 = 1261 \quad & \underline{1261} \\
\text{T.D.} = 3\,k^2 = 3\,(21)^2 = 1323 \quad & 564.170 \\
\text{C.D.} = 3\,(21)^2 + 3\,(21)(.4) + (.4)^2 = 1348.36 & \underline{539.344} \\
 & 24.826
\end{array}$$

Check. $(21.4)^3 = 9825.17 - 24.826$ (cf. Note 2, p. 181).

EXERCISE LXXXII

Extract the cube root of each of the following numbers:

1. 1728.
2. 571787.
3. 31855.013.
4. 148877.
5. 39304.
6. 426.957777.
7. 75.686967.
8. 34.7, to two decimal places.
9. .04, to two decimal places.
10. $3\frac{7}{8}$, to two decimal places.

121. Transformation of indicated roots.* From the definition of the symbol $\sqrt[n]{a}$ (§ 113) it follows that, whatever the values of h and k,

$$\sqrt{h^2k} = h\sqrt{k}: \tag{1}$$

for

$$\begin{aligned}(h\sqrt{k})^2 &= h\sqrt{k}\cdot h\sqrt{k}\\ &= hh\cdot\sqrt{k}\sqrt{k}\\ &= h^2k, \qquad [\text{since } \sqrt{k}\sqrt{k} = k]\end{aligned}$$

i.e.,

$$(h\sqrt{k})^2 = h^2k,$$

and $h\sqrt{k}$ is, therefore, a square root of h^2k.

Equation (1), read forward, tells *how to simplify an indicated square root of a number which contains a square factor;* and read backward, it tells *how to insert a coefficient under*

---

* Omit § 121 if radicals are to be studied before quadratics: see Preface.

a square root sign. Let the pupil translate this equation into verbal language, reading it both ways.

It also follows from the definition of $\sqrt[n]{a}$ (§ 113) that

$$(\sqrt{-x})^2 = -x. \tag{2}$$

Equations (1) and (2) will be useful in Chapter XII.

EXERCISE LXXXIII

Simplify the following expressions [cf. Eq. (1), § 121]:

1. $\sqrt{12}$.
2. $\sqrt{50}$.
3. $\sqrt{48}$.
4. $\sqrt{63}$.
5. $\sqrt{4a^2b}$.
6. $\sqrt{54x^3y^2}$.
7. $\sqrt{\frac{8}{9}}\left[i.e., \sqrt{\frac{4}{9}\cdot 2}\right]$.
8. $\sqrt{\frac{3}{4}}$.
9. $\sqrt{\frac{18}{25}}$.
10. $\sqrt{\frac{2}{3}}\left[i.e., \sqrt{\frac{6}{9}}\right]$.
11. $\sqrt{\frac{5}{8}}$.
12. $\sqrt{\frac{1}{2}a^3x}$.
13. $\sqrt{\frac{7}{8a}}\left[i.e., \sqrt{\frac{14a}{16a^2}}\right]$.
14. $\sqrt{\frac{3n^4}{2s^2t}}$.

Insert the following coefficients under the radical signs:

15. $3\sqrt{2}$.
16. $5\sqrt{7}$.
17. $-4\sqrt{5}$.
18. $7\sqrt{10}$.
19. $\frac{1}{2}\sqrt{8}$.
20. $-\frac{2}{3}\sqrt{9}$.
21. $\frac{3}{7}\sqrt{14}$.
22. $a\sqrt{2x}$.
23. $-5c\sqrt{b}$.
24. $\frac{1}{2}a\sqrt{b^2-4ac}$.

Expand the following expressions, and unite like terms:

25. $(3+\sqrt{5})^2$.
26. $(3+\sqrt{-5})^2$.
27. $(-1+\sqrt{3})^2$.
28. $(6-\sqrt{-2})^2$.
29. $(b-\sqrt{b^2-4ac})^2$.
30. $(b-\sqrt{4ac-b^2})^2$.
31. $[\frac{1}{2}(3-2\sqrt{5})]^2$.
32. $\left[\frac{-b+\sqrt{b^2-4ac}}{2a}\right]^2$.

CHAPTER XII *

QUADRATIC EQUATIONS (Elementary)

I. EQUATIONS IN ONE UNKNOWN NUMBER

122. Definitions. A **quadratic equation** has already been defined (§ 93) as an equation which, when simplified, is of the second degree in the unknown number or numbers. Thus, $3s^2 - 4 = 7s$, $ax^2 + bx + c = 0$, and $3m = 4m^2$ are quadratic equations in s, x, and m, respectively.

By transposing and uniting terms every quadratic in x, say, may evidently be reduced to the **standard form**

$$ax^2 + bx + c = 0,$$

wherein a, b, and c represent known numbers, and are usually called the **coefficients** of the equation. The term free from x, viz., c, is called the **absolute** (also **constant**) term.

E.g., by transposing, etc., $2x^2 + 5 - 3x = 7x - 8$ becomes $2x^2 - 10x + 13 = 0$: hence, for this particular equation $a = 2$, $b = -10$, and $c = 13$. Similarly, $6x^2 - 2x = 3x^2 - 4 - 2x$ becomes $3x^2 + 4 = 0$; here $a = 3$, $b = 0$, and $c = 4$.

An equation of the form $ax^2 + c = 0$ is often called a **pure** quadratic, while one containing both the first and second powers of the unknown number is called an **affected** quadratic.

123. Solution of pure quadratics. All pure quadratic equations may be solved like Exs. 1 and 2 below.

Ex. 1. Given $3x^2 - 12 = 0$; to find x.

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* Chapter XII may, if the teacher prefers, be omitted until Chapters XIV and XV have been studied: see Preface.

SOLUTION. On dividing through by 3, and transposing, the given equation becomes

$$x^2 = 4,$$

whence $\quad x = \pm 2,$*

i.e., $\quad x = 2$ or $x = -2$;

and each of these values is found to check.

Ex. 2. Solve the equation $5 - \dfrac{45 s^2}{16} = 0.$

SOLUTION. On dividing the given equation through by 5, clearing of fractions, and transposing, we obtain

$$9 s^2 = 16;$$

whence $\quad 3 s = \pm 4,$

i.e., $\quad s = \frac{4}{3}$ or $s = -\frac{4}{3}$;

and each of these values is found to check.

EXERCISE LXXXIV

Solve and check:

3. $4 x^2 = 36.$

4. $x^2 = \frac{1}{25}.$

5. $1 - x^2 = -48.$

6. $\dfrac{3 x^2}{4} = 27.$

7. $\dfrac{25}{2 y} - 8 y = 0.$

8. $4(v^2 + 3) = 32.$

9. $x^2 + 3 x = 3(x + 11) - 11.$

10. $\dfrac{r^2}{9} = \dfrac{(r+3)(r-3)}{8}.$

11. $3 cx^2 - 108 c^3 = 0.$

12. $(a+1)^2 x^2 = 4 a^2.$

13. $(k-6)^2 = 72 - 12 k.$

14. $x(x-5) = 2 - 5 x.$

15. $x(x+1) + 3 x^2 = x + \frac{9}{4}.$

16. $(r-3)^2 = 25.$

17. $(u + \frac{1}{5})^2 - \frac{49}{100} = 0.$

18. $(x-a)^2 = 9 b^2.$

19. $4 x^2 - 1 = a^2 + 2 a.$

20. $\dfrac{x}{16}(x-4) = \left(1 - \dfrac{x}{8}\right)^2.$

---

* Using the double sign in *each* member here gives $\pm x = \pm 2$,

i.e., either $x = 2$, (1) or $-x = 2$, (3)

or $x = -2$, (2) or $-x = -2$. (4)

But (3) and (4) give the same values of x as (2) and (1), respectively; hence in solving such an equation as $x^2 = 4$, the double sign need be used in one member only.

21. If x and y stand for unknown numbers, tell which of the following equations are simple, and which quadratic (cf. § 93):

$$a^4x^2 + a^2x + a = 0;\quad \frac{x-4}{2} = \frac{5}{x};\quad 5x - 7y = 11;\quad 5x + xy - 7y = 11;$$

$$2(x^2 - x) + 6 = 2x^2;\quad \frac{x}{y} - 4 = 5 + \frac{3}{y+2};\quad 3x + 4a^2 - 2ax = 7.$$

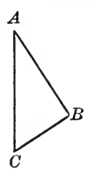
22. Reduce $5x^2 + 2 - 8x = 4(8 - x)$ to the "standard form." What are its coefficients? What is its absolute term?

23. Are the equations in Exs. 3–14 pure or affected? Explain. What is the absolute term in Ex. 9?

24. Show that the equation of Ex. 16 is a pure quadratic in $r - 3$ but an affected quadratic in r.

Solve each of the following equations for each letter it contains:

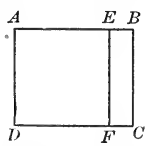
25. $s = \frac{1}{2}gt^2$. **26.** $\frac{1}{s} = \frac{2g}{v^2}$. **27.** $E = \frac{MV^2}{2}$ **28.** $R = K\frac{l}{d^2}$.



29. The square on the hypotenuse (longest side) of a right-angled triangle equals the sum of the squares on the other two sides. In the right-angled triangle ABC, the hypotenuse $AC = 5$, while $BC = \frac{5}{8} AB$; find AB and BC.

30. The area of a square is 169 square inches; find the perimeter and the diagonal of the square.

31. How many rods of fence will inclose a square garden whose area is $2\frac{1}{2}$ acres?



32. From a rectangular field, $ABCD$, whose width is $\frac{4}{5}$ of its length, there is cut off a square field, $AEFD$, whose area is 10 acres. Find the area of the rectangular field.

33. The surface area of a cube is 150 square inches. Find one edge and also the volume of the cube.

124. Quadratics solved by factoring. A quadratic equation may often be easily solved by reducing it to standard form (§ 122), and then factoring its first member (§ 72).

Ex. 1. Solve the equation $3x^2 + 4 = x^2 - 2x + 16$.

Solution. On transposing, uniting, and dividing through by 2, the given equation becomes

$$x^2 + x - 6 = 0,$$

i.e.,

$$(x-2)(x+3) = 0.$$

Now, as in § 72, this last equation is satisfied when

$$x - 2 = 0 \qquad \text{or} \qquad x + 3 = 0,$$

i.e., when $x = 2$ or when $x = -3$;

and each of these values of x, when substituted in the given equation, is found to check.

Ex. 2. Solve the equation $x(x-3) + x + 2 = 2(1 - x^2)$.

Solution. On transposing, etc., the given equation becomes

$$3x^2 - 2x = 0,$$

i.e.,

$$x(3x-2) = 0,$$

whence

$$x = 0 \text{ or } \frac{2}{3};$$

and these values of x are found to check.

EXERCISE LXXXV

Solve the following equations by factoring, and check the roots in each case:

3. $y^2 - 5y - 24 = 0$.
4. $m^2 - 16 = 0$.
5. $x^2 + 5x = 21 + x$.
6. $5x = x^2 - 14$.
7. $5s^2 = 8s$.
8. $2v^2 - 30 = 9v - v^2$.
9. $2x^2 - x = 3$.
10. $4x^2 = 9$.
11. $2c^2 = x(x + c)$.
12. $(4x)^2 = 14(4x) - 33$.
13. $22x + 3x^2 = 4x^2 - 48$.
14. $\dfrac{9 - 5s^2}{4} = 3s$.
15. $x^2 - 4x = 117$.
16. $13y + 2y^2 = 5y + 4y^2$.
17. $2x^2 - 20x = x^2 - 51$.
18. $3a(3a - 1) = 3a + 24$.
19. $2y^2 - 7y + 3 = 0$.
20. $x(x + m) = n(-x - m)$.
21. $-140 + x^2 = -23x$.
22. $r^2 - 5r = 5r - 25$.
23. $lx^2 - lkx + km = mx$.
24. $5t^2 - 3 = 10t - 3t^2$.
25. $ax^2 + bx = cx$.
26. $\dfrac{x^2}{bc} = \dfrac{x}{b} - \dfrac{x}{c} + 1$.

27. If a quadratic equation in one unknown number has no absolute term, show that one root of the equation must be zero.

125. Completing the square. What must be added to $x^2 + 6x$ to make it the square of $x + 3$? What must be added to $m^2 - 14m$ to make it the square of $m - 7$?

Since $(x \pm k)^2 = x^2 \pm 2kx + k^2$, therefore the expression $x^2 \pm 2kx$, whatever the value of k, lacks only the term k^2 of being the square of $x \pm k$; hence, *if the square of half the coefficient of the first power of* x *be added to an expression of the form* $x^2 + bx$, *the result will be an exact square.*

Such an addition is usually spoken of as **completing the square.**

E.g., if $(\frac{5}{2})^2$ is added to $y^2 + 5y$ it becomes $(y + \frac{5}{2})^2$.

126. Solution of quadratics by completing the square.* There are many quadratic equations which cannot easily be solved by the method of factoring given in § 124. All quadratic equations, however, may be solved by the method of completing the square, which is illustrated below.

Ex. 1. Solve the equation $2x^2 - 3 - 5x = 7x + 11$.

SOLUTION. On transposing, etc., this equation becomes

$$x^2 - 6x = 7.$$

Now, adding 9 to each member (§ 125, and Ax. 1), we obtain

$$x^2 - 6x + 9 = 16,$$

i.e., $(x - 3)^2 = 16,$

whence (§ 123) $x - 3 = \pm 4,$

i.e., $x - 3 = 4$ or $x - 3 = -4,$

and therefore $x = 7$ or -1.

Moreover, these values check, and are, therefore, the required roots.

* For the solution of quadratic equations by means of a formula, see § 178.

Ex. 2. Solve the equation $x^2+11x+1=8x$.

SOLUTION. On transposing, the given equation becomes

$$x^2+3x=-1,$$

whence, adding $(\frac{3}{2})^2$, $\quad x^2+3x+(\frac{3}{2})^2=-1+(\frac{3}{2})^2,$ [§ 125

i.e., $$(x+\tfrac{3}{2})^2=\tfrac{5}{4},$$

and hence $\quad x+\frac{3}{2}=\pm\sqrt{\frac{5}{4}}=\pm\frac{1}{2}\sqrt{5},$ [§ 121

i.e., $$x=-\frac{3}{2}\pm\frac{1}{2}\sqrt{5}=\frac{-3\pm\sqrt{5}}{2}.$$

Moreover, these values of x, viz., $\frac{-3+\sqrt{5}}{2}$ and $\frac{-3-\sqrt{5}}{2}$, check (cf. Ex. 25, p. 189), and are, therefore, the required roots.

EXERCISE LXXXVI

3. Solve the equation $ax^2+bx+c=0$.

SOLUTION. On transposing and dividing by a, this equation becomes

$$x^2+\frac{b}{a}x=-\frac{c}{a},$$

whence $\quad x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2},$ [§ 125

i.e., $$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2};$$

therefore $\quad x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{\pm\sqrt{b^2-4ac}}{2a},$ [§ 121

i.e., $$x=-\frac{b}{2a}+\frac{\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Moreover, these values of x, viz., $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$, check (cf. Ex. 29, p. 189) and are, therefore, roots of the given equation.

4. What must be added to each of the following expressions in order to complete the square (cf. § 125): x^2+8x; P^2-5P; $(x+y)^2-4(x+y)$?

5. How do we find the number which added to r^2+ar completes the square? Explain.

Solve the following equations by the method of completing the square, and check the roots in each case:

6. $m^2 - 6m = 40.$
7. $y^2 - 10y = 75.$
8. $15 = 2x + x^2.$
9. $-8 = 2x^2 + 10x.$
10. $x^2 = x + \frac{3}{4}.$
11. $3x^2 - 2x = 1.$
12. $2x^2 + 3 = 7x.$
13. $3x^2 - 10 = 7x.$
14. $(2y - 3)^2 = 6y + 1.$
15. $m(m + 4) = 7.$
16. $a^2 - 6a + 10 = 0.$
17. $3(v^2 - v) = 2v^2 + 5v + 4.$
18. $y^2 - 2cy = 1.$
19. $r^2 + 2ar = d.$
20. $x^2 - 3x - 2 = 0.$
21. $x^2 - 3x + 4 = 0.$
22. $\frac{5}{3}c^2 - 7c = c(c + 1).$
23. $6 + 5t = 6t^2.$
24. $12x^2 - x = 6.$
25. $s^2 = \frac{4}{3}s + 2.$
26. $r^2 - er = f.$
27. $3x^2 + 5x - 7 = x^2 - 2x.$
28. $8m - 10 = 3m^2.$
29. $\frac{1}{2}x - \frac{3}{4}x^2 + 2 = 0.$
30. $(y - 3)^2 - 4(y - 3) = 117.$
31. $(2m - 3)^2 - 6(m + 1) + 8 = 0.$
32. $c^2x^2 + 2dx = -e.$
33. $(n + 1)^2 - 8(n + 1) = 16.$

34. Write a carefully worded rule for solving such quadratic equations as those in Exs. 6–33 above.

35. Show that the rule asked for in Ex. 34 will serve to solve such an equation as $x^2 + 6x = 0$. Is this equation more easily solved by completing the square or by factoring?

127. Avoiding fractions in completing the square. The method employed in § 126 for completing the square often introduces fractions into the work, and these sometimes become troublesome (cf. Exs. 2 and 3, p. 195). A method which avoids fractions is illustrated below.

Ex. 1. Solve the equation $5x^2 - 6x = -1.$

SOLUTION. On multiplying through by 5, we obtain

$$25x^2 - 30x = -5,$$

i.e.,
$$(5x)^2 - 6(5x) = -5,$$

whence, adding 9,
$$(5x)^2 - 6(5x) + 9 = 4,$$

i.e.,
$$(5x - 3)^2 = 4;$$

therefore $$5x-3=\pm 2,$$
from which $$5x=3\pm 2=5 \text{ or } 1,$$
i.e., $$x=1 \text{ or } \tfrac{1}{5};$$
and each of these values is, on substitution, found to check.

Ex. 2. Solve the equation $ax^2+bx=-c$.

Solution. On multiplying through by $4a$, we obtain
$$4a^2x^2+4abx=-4ac,$$
i.e., $$(2ax)^2+2b(2ax)=-4ac,$$
whence, adding b^2, $$(2ax)^2+2b(2ax)+b^2=b^2-4ac,$$
i.e., $$(2ax+b)^2=b^2-4ac;$$
therefore $$2ax+b=\pm\sqrt{b^2-4ac},$$
from which $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ [cf. Ex. 3, p. 195.

Note. From the above solutions we see that, if an equation of the form $ax^2+bx+c=0$ is multiplied through by a or $4a$, according as b is even or odd, fractions can be avoided in the solution.

EXERCISE LXXXVII

3. What must be added to each of the following expressions in order to complete the square: $4x^2+8x$; $9m^4+12m^2$; $25c^2d^2-10cd$; and $4c^2m^2-4cm$?

In each of Exs. 4–11 below, (1) name the factor by which both members of the equation must be multiplied if fractions are to be avoided in completing the square; (2) solve the equation by the method of § 127.

4. $5x^2+6x=8$.

5. $3y^2+4y=95$.

6. $2m^2+3m=27$.

7. $2t^2+7t+6=0$.

8. $3z^2=2+5z$.

9. $7=2x+3x^2$.

10. $5x^2-x-3=0$.

11. $15x^2-7x-2=0$.

12. By the method of § 127, solve Exs. 12–14 and 20–25, p. 196.

Solve the following equations by the method of § 127:

13. $\dfrac{t^2+3t+5}{2}=-\dfrac{t+1}{3}.$

14. $mx^2-6x+3=0.$

15. $x^2+px+q=0.$

16. $mx^2=2nx-k.$

17. $\dfrac{u^2+u}{3}+\dfrac{u^2+\frac{1}{3}}{4}=0.$

18. $3y^2-4ky+2=0.$

19. $mx^2+nx+p=0.$

20. $ax^2-5ax=a-11.$

128. Fractional equations which lead to quadratics. As in § 97, so here, we first clear the given equation of fractions, then solve the resulting integral equation, and finally check the results so as to guard against the introduction of extraneous roots (§ 97, note).

Ex. 1. Solve the equation $\dfrac{x+5}{x+2}+1=3x.$

SOLUTION. On clearing of fractions, etc., this equation becomes

$$3x^2+4x-7=0,$$

whence, solving as in § 127, we obtain

$$x=1 \text{ or } -\tfrac{7}{3};$$

and each of these values, when substituted in the given equation, is found to check.

Ex. 2. Solve the equation $\dfrac{x}{1-x}+\dfrac{4x+3}{x+1}=\dfrac{2x^2}{x^2-1}.$

SOLUTION. On clearing of fractions, etc., we obtain

$$x^2-2x-3=0,$$

whence $x=3 \text{ or } -1,$ [§ 126

of which 3 checks, but -1 is extraneous (§ 97, note).

EXERCISE LXXXVIII

Solve the following fractional equations, being careful to exclude all extraneous roots:

3. $15x+\dfrac{2}{x}=11.$

4. $\dfrac{1}{x}-2+x=\dfrac{2}{x}.$

5. $\dfrac{2x-2}{5x+5}=\dfrac{x-1}{x+1}.$

6. $\dfrac{1}{1-s}+\dfrac{1}{1+s}=\dfrac{8}{3}.$

7. $\dfrac{3}{2(x^2-1)} - \dfrac{1}{4(x+1)} = \dfrac{1}{8}$.

8. $\dfrac{x-2}{x+2} + \dfrac{x+2}{x-2} = 2\left(\dfrac{x+3}{x-3}\right)$.

9. $\dfrac{1}{x-1} + \dfrac{1}{x-2} = \dfrac{1}{3-x}$.

10. $\dfrac{20}{s+3} + \dfrac{40}{s^2+4s+3} = -\dfrac{3s+7}{s+1}$.

11. $\dfrac{2y+1}{1-2y} - \dfrac{5}{7} = \dfrac{y-8}{2}$.

12. $\dfrac{2a+x}{2a-x} + \dfrac{a-2x}{a+2x} = 2\frac{2}{3}$.

13. $\dfrac{bx}{a-x} + b = \dfrac{a(x+2b)}{a+b}$.

14. $\dfrac{z}{z-1} - \dfrac{z}{z+1} = c$.

15. $\dfrac{3x}{x+5} + \dfrac{42}{(x+5)(x-2)} = 8 + \dfrac{6}{x-2}$.

129. Problems which lead to quadratics. As in § 50, so here, the important steps in the solution of a problem are:

1. To translate the verbal language of the problem into algebraic language, *i.e.*, into equations.

2. To solve these equations.

3. To check, and interpret, the results.

Special emphasis should be laid upon testing and interpreting results: a *problem* often contains restrictions upon its numbers, expressed or implied, which are not translated into the *equations*, hence the solutions of the equations may or may not be solutions of the problem itself (cf. § 98).

Prob. 1. A farmer purchased some sheep for \$168, and later sold all but four of them for the same sum. If his profit on each sheep sold was \$1, how many sheep did he buy?

SOLUTION

Let $x =$ the number of sheep purchased.

Then $\dfrac{168}{x} =$ the number of dollars each sheep cost,

and $\dfrac{168}{x-4} =$ the number of dollars received for each sheep,

and hence $\dfrac{168}{x-4} - \dfrac{168}{x} = 1$, [Profit on each sheep being \$1

therefore (§ 128) $x = 28$ or -24.

The first of these values, viz., 28, is found to be a solution of the problem as well as of the equation, but while the second satisfies the equation it cannot satisfy the problem, since the number of sheep purchased is necessarily a positive integer.

Prob. 2. At a certain dinner party it is found that 6 times the number of guests exceeds the square of $\frac{2}{3}$ their number by 8; how many guests are there?

SOLUTION

Let $x =$ the number of guests.

Then the expressed condition of the problem is

$$6x - \left(\frac{2x}{3}\right)^2 = 8,$$

i.e., $$2x^2 - 27x + 36 = 0,$$

whence $$x = 12 \text{ or } \tfrac{3}{2}.$$

Here, too, an *implied* condition of the problem is that the answer must be a positive integer; hence $\frac{3}{2}$, although it satisfies the equation, it is not a solution of the problem.

Prob. 3. The sum of the ages of a father and his son is 100 years, and one tenth of the product of the numbers of years in their ages, minus 180, equals the number of years in the father's age; what is the age of each?

SOLUTION

Let $x =$ the number of years in the father's age.

Then $100 - x =$ the number of years in the son's age,

and the condition of the problem states that

$$\frac{x(100-x)}{10} - 180 = x,$$

whence $$x = 60 \text{ or } 30.$$

Although both 60 and 30 are positive integers, yet 30 is not a solution of the problem: it would make the son older than the father. Hence the father is 60 years old, and the son 40.

If, in the above problem, "two persons" be substituted for "a father and his son," etc., then both solutions are admissible, and the ages are either 60 and 40 years, or 30 and 70 years.

EXERCISE LXXXIX

4. Divide 10 into two parts whose product is $22\frac{3}{4}$.

5. Find two numbers whose difference is 11, and whose sum multiplied by the greater is 513.

6. A man bought a flock of sheep for \$ 75. If he had paid the same sum for a flock containing 3 sheep more they would have cost him \$ 1.25 less per head. How many did he buy?

Is each solution of the equation of this problem a solution of the problem itself? Explain.

7. A clothier having bought some cloth for \$30 found that if he had received 3 yards more for the same money, the cloth would have cost him 50 cents less per yard. How many yards did he buy? Has this problem more than one solution?

8. Find two numbers whose sum is 10 and whose product is 42. Can these numbers be *real* (see Note, § 114)?

9. Find two consecutive integers the sum of whose squares is 61. How many solutions has the equation of this problem? Show that each of these is a solution of the problem also.

10. Are there two consecutive *integers* the sum of whose squares is 118? Are there two *numbers* whose difference is 1, and the sum of whose squares is 118? What are they? How does the second of these questions differ from the first?

11. Find three consecutive integers whose sum is equal to the product of the first two.

12. Is it possible to find three consecutive integers whose sum equals the product of the first and last? How is the impossibility of such a set of numbers shown?

13. In selling a yard of silk at 75 cents, a merchant gains as many per cent as there are cents in its cost. Find the cost.

14. A cow staked out to graze can graze over a circle 616 square feet in area; how long is the rope by which she is tied? [The area of a circle of radius r is $\pi \cdot r^2$; $\pi = 3\frac{1}{7}$, approximately.]

15. Two circles are such that the difference of their radii is 3 inches, and the sum of their areas $279\frac{5}{7}$ sq. in. Find the radius of each circle.

16. Find two numbers whose sum is $\frac{5}{6}$, and whose difference is equal to their product. How many solutions has this problem?

17. The product of three consecutive integers is divided by each of them in turn, and the sum of the three quotients is 74. What are these integers? How many solutions has this problem? Explain.

18. If the product of two numbers is 6, and the sum of their reciprocals is $\frac{35}{36}$, what are the numbers? How many solutions has the equation of this problem? How many solutions has the problem itself? Explain.

19. A merchant who had purchased a quantity of flour for $96 found that if he had obtained 8 barrels more for the same money, the price per barrel would have been $2 less. How many barrels did he buy? How many solutions has this problem? Explain.

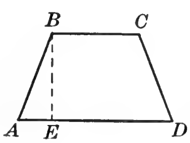
20. Why is it that the solutions of the equation of a problem are not always solutions of the problem itself? (Cf. § 129.)

21. In a rectangle whose area is $55\frac{1}{4}$ sq. in., the sum of the length and breadth is 15 in.; find the length.

22. Find the length of a rectangle whose area is 464 sq. in., and the sum of whose length and breadth is 16 in.

Interpret the imaginary result in this problem (cf. § 98). Does an imaginary result *always* show that the conditions of the problem are impossible of fulfillment (cf. Prob. 8, p. 201)?

23. The number of square inches in the surface area of a cube exceeds the number of cubic inches in its volume by 8 times the number of inches in one edge. Find the edge of the cube. How many solutions has this problem? Explain.



24. In the trapezoid $ABCD$, whose area is 75 sq. in., the altitude BE equals BC, and AD is 5 in. longer than BC; find BC and AD. [The area of $ABCD = \frac{1}{2}(BC + AD) \cdot BE$.]

25. A triangle whose base is 2 in. longer than its altitude has an area equal to that of a rectangle 10 in. by 4 in. Find the base and altitude of the triangle. [The area of a triangle equals half the product of its base and altitude.]

26. A boating club on returning from a short cruise found that its expenses had been $90, and that the number of dollars each member had to pay was less by $4\frac{1}{2}$ than the number of members in the club. How many members were there in the club?

27. If in Prob. 26 the expense of the cruise had been $145 the other conditions remaining unchanged, how many members would the club contain?

What is the significance of the fractional and negative results in this problem? Do such results always indicate that the conditions of a problem are impossible of fulfillment?

28. The number of miles in the distance between two cities is such that its square root, plus its half, equals 12. What is this distance? Has this problem more than one solution? Explain.

29. When a certain train has traveled 5 hours it is still 60 miles from its destination. If by traveling 5 miles faster per hour, it could make the entire trip in 1 hour less than the scheduled time, find the entire distance; also the actual speed.

30. The hypotenuse of a right-angled triangle is 10 inches, and one of the sides is 2 inches longer than the other; required the length of the sides (cf. Ex. 29, p. 192).

31. It took a number of men as many days to dig a trench as there were men. If there had been 6 more men, the work would have been done in 8 days. How many men were there?

32. A crew can row $5\frac{1}{2}$ miles downstream and back again in 2 hours and 23 minutes; if the rate of the current is $3\frac{1}{2}$ miles an hour, find the rate at which the crew can row in still water.

33. From a thread whose length is equal to the perimeter of a square, one yard is cut off, and the remainder is equal to the perimeter of another square whose area is $\frac{4}{9}$ that of the first. What was the length of the thread at first?

34. The diagonal and the longer side of a rectangle are together equal to five times the shorter side, and the longer side exceeds the shorter by 35 yards. Find the area of the rectangle.

35. A ladder 13 ft. long leans against a vertical wall. When the distance from the base of the wall to the foot of the ladder is 7 ft. less than the height of the wall, the ladder just reaches to the top of the wall. How high is the wall (cf. Prob. 30)?

36. If one train, by going 15 miles an hour faster than another, requires 12 minutes less than the other to run 36 miles, what is the speed of each train?

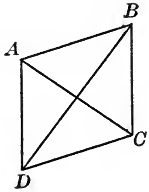
37. A tank can be filled by one of its two feed-pipes in 2 hours less time than by the other, and by both pipes together in $1\frac{7}{8}$ hours. In what time can each pipe separately fill the tank?

38. The owner of a lot 56 rods long and 28 rods wide divided it into 4 equal rectangular lots, by constructing through it two streets of uniform width. If these streets decrease the available area of the lot by 2 acres, what is their width?

39. One of two casks contains twice as many gallons of water as the other does of wine; 6 gallons are drawn from each cask, exchanged, and emptied into the other; it is then found that the percentage of wine in each cask is the same. How many gallons of water did the first cask originally contain?

40. A and B together can do a given piece of work in a certain time; but if they each do one half of this work separately, A works one day less, and B two days more, than when they work together. In how many days can they do the work together?

41. In going a mile, the hind wheel of a carriage makes 145 revolutions less than the front wheel, but if the hind wheel were 16 inches greater in circumference, it would then make 200 revolutions less than the front wheel. What is the circumference of the front wheel?



42. In the figure, $AB = BC = CD = DA = 10$ inches, the diagonals AC and DB bisect each other at right angles, and DB is 4 inches longer than AC. Find the lengths of AC and DB, and the area of the figure.

130. Equations in quadratic form. Equations which contain only *two* different powers of the unknown number, one of these powers being the square of the other, are said to be **in quadratic form.** Thus, $x^6 + 7x^3 = 8$, $au^{2n} + bu^n + c = 0$, and $(2s^3+1)^2 - 5(2s^3+1) = 4$ are in quadratic form.

Such equations may be solved as follows:

Ex. 1. Solve the equation $2x^2(x^2+1) = 5 - x^2$.

Solution. When simplified, the given equation becomes

$$2x^4 + 3x^2 - 5 = 0,$$

or, putting y for x^2, $\quad 2y^2 + 3y - 5 = 0,$

whence $\quad y = 1 \text{ or } -\frac{5}{2},$ [§ 126

i.e. (since $y = x^2$), $\quad x^2 = 1 \text{ or } -\frac{5}{2},$

whence $\quad x = \pm 1 \text{ or } \pm\sqrt{-\frac{5}{2}}.$

Moreover, each of these values $(1, -1, \sqrt{-\frac{5}{2}}, \text{ and } -\sqrt{-\frac{5}{2}})$ checks (§ 121), and is therefore a root of the given equation.

Ex. 2. Solve the equation $\sqrt{x^2-5x+10} = 2x^2 - 10x + 14$.

Solution. This equation may be written thus:

$$\sqrt{x^2-5x+10} = 2(x^2-5x+10) - 6;$$

and, on putting y for $\sqrt{x^2-5x+10}$, the given equation becomes

$$y = 2y^2 - 6,$$

whence $\quad y = 2 \text{ or } -\frac{3}{2},$ [§ 126

i.e., $\quad \sqrt{x^2-5x+10} = 2 \text{ or } -\frac{3}{2},$

and therefore $\quad x^2 - 5x + 10 = 4 \text{ or } \frac{9}{4},$ [Squaring

whence $\quad x = 2,\ 3,\ \dfrac{5+\sqrt{-6}}{2}, \text{ or } \dfrac{5-\sqrt{-6}}{2},$ [§ 126

all of which values check (cf. Ex. 28, p. 189), and are therefore the required roots.

EXERCISE XC

Solve, and check as the teacher directs:

3. $m^4 - 16 = 0.$

4. $x^4 - 8x^2 + 12 = 0.$

5. $y^4 - 25y^2 + 144 = 0.$

6. $n^4 = 18n^2 - 32.$

7. $y - \sqrt{y} = 6.$

8. $x = 3 - 2\sqrt{x}.$

9. $20x^4 - 23x^2 = -6.$

10. $4 - m^2 = 18m^4.$

11. $13\sqrt{z} - 5 = 6z.$

12. $x^2 + \dfrac{1}{x^2} = a^2 + \dfrac{1}{a^2}.$

13. $(x^2+1)^2 + 4(x^2+1) = 45.$

14. $\left(\dfrac{12}{u} - 1\right)^2 + 8\left(\dfrac{12}{u} - 1\right) = 33.$

15. $x - 2 = \sqrt{x-2} + 6.$

16. $(m+1) - 5\sqrt{m+1} = 6.$

17. $2s - 3 = 7\sqrt{2s-3} - 12.$

18. $2k^2 - 6\sqrt{2k_2 - 1} = 8.$

Hint. Write equation thus:

$$(2k^2 - 1) - 6\sqrt{2k^2 - 1} = 7.$$

19. $x^2 - x + \sqrt{x^2 - x - 3} = 9.$

20. $5x^2 + 2\sqrt{5x^2 - x} = 8 + x.$

21. $\dfrac{m^2+2}{3} + \dfrac{3}{m^2+2} = 2.$

Hint. Let $x = \dfrac{m^2+2}{3}$, then $\dfrac{1}{x} = ?$

22. $\dfrac{v^2}{v+1} - \dfrac{v+1}{v^2} = \dfrac{7}{12}.$

23. $\dfrac{r^2+6}{r} + \dfrac{10r}{r^2+6} = 7.$

24. $\dfrac{y+2}{y^2+4} + \dfrac{2(y^2+4)}{y+2} = \dfrac{51}{5}.$

25. $5\sqrt{m^2 - 10m + 42} = m^2 - 10m + 6.$

26. $t^2 - 7t + \sqrt{t^2 - 7t + 18} = 24.$

27. $x^4 + 4x^3 - 8x + 3 = 0.$

Hint. By extracting the square root of the first member this equation may be written in the form $(x^2 + 2x - 2)^2 = 1$. (Cf. Ex. 32, p. 183.)

28. $y^4 + 2y^3 + 5y^2 + 4y = 60.$

29. $16x^4 - 8x^3 - 31x^2 + 8x + 15 = 0.$

30. $9x^4 + 6x^3 - 83x^2 - 28x + 147 = 0.$

II. SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS

[Two Unknown Numbers]

131. One equation simple and the other quadratic. When one equation is simple and the other quadratic, we may always eliminate by substitution [§ 103 (i)].

Ex. 1. Solve the following system of simultaneous equations:

$$\begin{cases} 3x - 2y = 3, & (1) \\ x^2 + 4y^2 = 13. & (2) \end{cases}$$

SOLUTION. From Eq. (1), $$x=\frac{3+2y}{3}, \tag{3}$$
whence, by substituting this value of x, Eq. (2) becomes
$$\left(\frac{3+2y}{3}\right)^2+4y^2=13, \tag{4}$$
and, on expanding and simplifying, Eq. (4) becomes
$$10y^2+3y-27=0,$$
whence (§ 126) $$y=\tfrac{3}{2} \text{ or } -\tfrac{9}{5}.$$
Substituting these values of y in Eq. (3) shows that
$$x=2 \text{ or } -\tfrac{1}{5},$$
according as $$y=\tfrac{3}{2} \text{ or } -\tfrac{9}{5}.$$
Moreover, these pairs of values, viz., $\begin{cases} x=2, \\ y=\frac{3}{2}, \end{cases}$ and $\begin{cases} x=-\frac{1}{5}, \\ y=-\frac{9}{5}, \end{cases}$
satisfy the given system, and are therefore the solutions sought.

EXERCISE XCI

Solve the following systems of equations and check your results:

2. $\begin{cases} x^2-y^2=80, \\ x+y=10. \end{cases}$

3. $\begin{cases} x^2+xy=12, \\ x-y=2. \end{cases}$

4. $\begin{cases} 3uv-v=10u, \\ u+2=v. \end{cases}$

5. $\begin{cases} 4x+3y=9, \\ 2x^2+5xy=3. \end{cases}$

6. $\begin{cases} (x+3)(y-7)=48, \\ x+y=18. \end{cases}$

7. $\begin{cases} st=42, \\ s-t=19. \end{cases}$

8. $\begin{cases} \dfrac{x}{2}+\dfrac{5y}{4}=10, \\ \dfrac{xy}{3}-\dfrac{y^2}{6}=4. \end{cases}$

9. $\begin{cases} 2s+3t=10, \\ t(s+t)=8. \end{cases}$

10. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{4}{15}, \\ x-y=-10. \end{cases}$

11. $\begin{cases} \frac{2}{3}x+\frac{3}{5}y=15, \\ 3x-\dfrac{5x}{y}=24. \end{cases}$

12. $\begin{cases} 1.5x-.5y=6, \\ .1x^2+.1xy=16. \end{cases}$

13. $\begin{cases} 16+4v+2u^2=5uv, \\ 11v-5u=4. \end{cases}$

14. $\begin{cases} \dfrac{3}{xy}+14=\dfrac{2}{x^2}+\dfrac{1}{y^2}, \\ \dfrac{2}{x}-\dfrac{1}{y}=7. \end{cases}$

132. Both equations quadratic: one homogeneous. An equation is said to be **homogeneous** if all of its terms are of the same degree in the unknown numbers (cf. §§ 34, 93).

If one of two given quadratic equations is homogeneous, the system may always be solved as follows:

Ex. 1. Solve the following system of equations:

$$\begin{cases} 6x^2 + 5xy - 6y^2 = 0, \\ 2x^2 - y^2 + 5x = 9. \end{cases} \qquad \begin{matrix} (1) \\ (2) \end{matrix}$$

Solution. On dividing Eq. (1) by y^2, it becomes

$$6\left(\frac{x}{y}\right)^2 + 5\left(\frac{x}{y}\right) - 6 = 0, \qquad (3)$$

whence (§ 126) $$\frac{x}{y} = \frac{2}{3} \text{ or } \frac{x}{y} = -\frac{3}{2}, \qquad (4)$$

i.e., $$x = \tfrac{2}{3}y \text{ or } x = -\tfrac{3}{2}y. \qquad (5)$$

On substituting the *first* of these two values of x, viz., $\frac{2}{3}y$, in Eq. (2), we obtain

$$2(\tfrac{2}{3}y)^2 - y^2 + 5(\tfrac{2}{3}y) = 9,$$

i.e., $$y^2 - 30y + 81 = 0,$$

whence (§ 124) $$y = 27 \text{ or } y = 3,$$

and, since $x = \frac{2}{3}y$, the *corresponding* values of x are 18 and 2.

Moreover, these pairs of values, viz., $\begin{cases} x = 18, \\ y = 27, \end{cases}$ and $\begin{cases} x = 2, \\ y = 3, \end{cases}$ are found to check, and are therefore solutions of the system.

Again, by substituting in Eq. (2) the *second* of the two values of x in Eq. (5), we find two other solutions of the given system viz. $\begin{cases} x = -\frac{9}{2}, \\ y = 3, \end{cases}$ and $\begin{cases} x = \frac{9}{7}, \\ y = -\frac{6}{7}. \end{cases}$

Note. The above method may be somewhat simplified by substituting a single letter, say v, for the fraction x/y in Eq. (3), *i.e.*, by putting $x = vy$ in the *homogeneous* equation. Thus, putting vy for x, Eq. (1) becomes

$$6v^2y^2 + 5vy^2 - 6y^2 = 0,$$

and hence, dividing by y^2, $$6v^2 + 5v - 6 = 0,$$

whence (§ 126) $$v = \tfrac{2}{3} \text{ or } v = -\tfrac{3}{2};$$

and, since $x = vy$, therefore $x = \frac{2}{3}y$ or $x = -\frac{3}{2}y$. From here on the work is the same as that already given.

EXERCISE XCII

2. Which of the equations in Ex. 3–12 below are homogeneous? Why? Write a homogeneous cubic equation involving the unknown numbers r and s.

By the method of Ex. 1 (or of the note) solve the following systems of equations; check your results as the teacher directs:

3. $\begin{cases} x^2+y^2=40, \\ 3x^2-10xy+3y^2=0. \end{cases}$

4. $\begin{cases} xy+y^2=28, \\ 4x^2-27xy+18y^2=0. \end{cases}$

5. $\begin{cases} x^2+xy-14=y-x, \\ 2x^2-3y^2=xy. \end{cases}$

6. $\begin{cases} t(s+t)=36, \\ 5s^2-19st+12t^2=0. \end{cases}$

7. $\begin{cases} 3u^2-5uv=2v^2, \\ u(u-v)=8. \end{cases}$

8. $\begin{cases} 5x^2-7xy-24y^2=0, \\ xy+2y^2=5. \end{cases}$

9. $\begin{cases} xy+3y^2-20=0, \\ 5x^2=13xy-6y^2. \end{cases}$

10. $\begin{cases} 8x^2+y^2=36, \\ 4x^2-9xy+5y^2=0. \end{cases}$

11. $\begin{cases} 2(x^2+y^2)=5xy, \\ x^2-y^2=75. \end{cases}$

12. $\begin{cases} 5x^2+4xy=y^2, \\ x^2+3x=5+y. \end{cases}$

133. Both equations homogeneous except for the absolute term. A system consisting of two quadratic equations each of which is *homogeneous in the terms containing the unknown numbers* can be solved by the method of § 132, Note.

Ex. 1. Solve $\begin{cases} 3x^2+3xy+2y^2=8, & (1) \\ x^2-xy-4y^2=2. & (2) \end{cases}$

Solution. By substituting vy for x in Eqs. (1) and (2) we obtain

$$3v^2y^2+3vy^2+2y^2=8, \tag{3}$$

and

$$v^2y^2-vy^2-4y^2=2, \tag{4}$$

whence, from (3) and (4), respectively,

$$y^2=\frac{8}{3v^2+3v+2} \text{ and } y^2=\frac{2}{v^2-v-4}; \tag{5}$$

therefore

$$\frac{8}{3v^2+3v+2}=\frac{2}{v^2-v-4}, \tag{6}$$

whence

$$v=-2 \text{ or } 9.$$

Hence, from (5), $y^2 = 1$ or $y^2 = \frac{1}{34}$,

i.e., $y = 1$ or -1 or $y = +\sqrt{\frac{1}{34}}$ or $-\sqrt{\frac{1}{34}}$,

and substituting these values of y and v in

$$x = vy,$$

we obtain, as *corresponding* values,

$$x = -2 \text{ or } +2, \text{ and also } +9\sqrt{\tfrac{1}{34}} \text{ or } -9\sqrt{\tfrac{1}{34}};$$

and, checking, we find the solutions of the given system to be:

$$\begin{cases} x = -2, \\ y = 1, \end{cases} \quad \begin{cases} x = 2, \\ y = -1, \end{cases} \quad \begin{cases} x = 9\sqrt{\frac{1}{34}}, \\ y = \sqrt{\frac{1}{34}}, \end{cases} \quad \begin{cases} x = -9\sqrt{\frac{1}{34}}, \\ y = -\sqrt{\frac{1}{34}}. \end{cases}$$

Note. The success of the "vy method" employed above is due to the fact that by eliminating the absolute term from the given system of equations we obtain a *homogeneous* equation (cf. § 132).

EXERCISE XCIII

Solve the following systems, and check your results:

2. $\begin{cases} x^2 + y^2 = 29, \\ xy = 10. \end{cases}$

3. $\begin{cases} m^2 - mn = 8, \\ mn + n^2 = 12. \end{cases}$

4. $\begin{cases} x^2 - 3xy = -14, \\ xy + 2y^2 = 39. \end{cases}$

5. $\begin{cases} y^2 + 2xy = -9, \\ x^2 - xy = 70. \end{cases}$

6. $\begin{cases} 2x^2 - xy = 28, \\ x^2 + 2y^2 = 18. \end{cases}$

7. $\begin{cases} y^2 + 15 = 2xy, \\ x^2 + y^2 = 21 + xy. \end{cases}$

8. $\begin{cases} x^2 + xy = 40, \\ 27 + 2y^2 - 3xy = 0. \end{cases}$

9. $\begin{cases} 2x^2 + 3xy + y^2 = 20, \\ 5x^2 + 4y^2 = 41. \end{cases}$

10. $\begin{cases} x^2 - xy - y^2 = 20, \\ x^2 - 3xy + 2y^2 = 8. \end{cases}$

11. $\begin{cases} u^2 + 3uv + v^2 = 61, \\ u^2 - v^2 = 31 - 2uv. \end{cases}$

12. $\begin{cases} \dfrac{3y^2}{x} - \dfrac{5y}{2} + \dfrac{2x}{3} = x - 2, \\ \dfrac{4x}{3} - \dfrac{4y^2}{x-1} = \dfrac{y^2 + 2xy}{2(1-x)}. \end{cases}$

13. Solve Ex. 1 by eliminating the absolute term and then applying § 132.

By the method suggested in Ex. 13, solve:

14. $\begin{cases} x^2 - 3xy + 4y^2 = 8y, \\ x^2 - 4xy + 2y^2 = 2y. \end{cases}$

15. $\begin{cases} x^2 - 4xy + 3y^2 = -3y, \\ 3x^2 - 5xy = 6y. \end{cases}$

16. $\begin{cases} 3x^2 - 5xy - 4y^2 = 3x, \\ 9x^2 + xy - 2y^2 = 6x. \end{cases}$

17. $\begin{cases} 4x^2 + 6xy - y^2 = \frac{1}{3}y, \\ 6x^2 - 9xy + 2y^2 = 2y. \end{cases}$

134. Special devices. The kinds of systems of equations specified in §§ 131–133 occur frequently, and, although they present themselves in a great variety of forms, they may *always* be solved by the methods there given.

Special devices for elimination, however, often give simpler and more elegant solutions; some of these devices are illustrated below.

(i) *Solving by first finding $x + y$ and $x - y$.*

Ex. 1. Solve the equations $\begin{cases} x - y = 5, & (1) \\ xy = -6. & (2) \end{cases}$

SOLUTION. From (1), $x^2 - 2xy + y^2 = 25,$ (3)

and from (2), $4xy = -24;$ (4)

adding (4) to (3), $x^2 + 2xy + y^2 = 1,$ (5)

whence $x + y = \pm 1;$ (6)

and from (1) and (6), $x = 3$ or $2.$

The *corresponding* values of y are $y = -2$ or $-3;$

i.e., the solutions of the given system are: $\begin{cases} x = 3, \\ y = -2, \end{cases}$ and $\begin{cases} x = 2, \\ y = -3. \end{cases}$

Ex. 2. Solve the equations $\begin{cases} x^2 + y^2 = 5, & (1) \\ x^2 - xy + y^2 = 3. & (2) \end{cases}$

SOLUTION. Subtracting (2) from (1), $xy = 2,$ (3)

whence, adding $2 \cdot (3)$ to (1), $x^2 + 2xy + y^2 = 9,$ (4)

and subtracting $2 \cdot (3)$ from (1), $x^2 - 2xy + y^2 = 1;$ (5)

therefore, from (4), $x + y = \pm 3,$ (6)

and from (5), $x - y = \pm 1;$ (7)

from (6) and (7), we now easily find the following solutions:

$$\begin{cases} x = 2, \\ y = 1, \end{cases} \quad \begin{cases} x = 1, \\ y = 2, \end{cases} \quad \begin{cases} x = -2, \\ y = -1, \end{cases} \quad \text{and} \quad \begin{cases} x = -1, \\ y = -2, \end{cases}$$

all of which check.

(ii) *Solving by dividing one equation by the other.*

Ex. 3. Solve the equations $\begin{cases} x^2 - y^2 = 3, & (1) \\ x - y = 1. & (2) \end{cases}$

SOLUTION. On dividing (1) by (2), member by member, we obtain $x + y = 3,$ (3)
whence, from (2) and (3), $x = 2$, and $y = 1$.

Ex. 4. Solve the equations $\begin{cases} x^3 + y^3 = 26, & (1) \\ x + y = 2. & (2) \end{cases}$

SOLUTION. On dividing (1) by (2), member by member, we obtain $x^2 - xy + y^2 = 13,$ (3)
and (2) and (3) may now be solved either like Exs. 1 and 2 above, or by the method of § 131.

(iii) *Solving by considerations of symmetry.* An equation is **symmetric** with regard to two of its letters if it is not changed by interchanging those letters. Thus: $x + y = 8$, and $s^2 + st + t^2 = 5(s + t)$ are symmetric equations.

Two equations which are symmetric (or symmetric except for the *signs* of one or more terms) may often be solved by substituting $u + v$ and $u - v$, respectively, for their unknown numbers.

Ex. 5. Solve the equations $\begin{cases} x^2 + y^2 = 6, & (1) \\ xy = 2(x + y) - 5. & (2) \end{cases}$

SOLUTION. On putting $x = u + v$ and $y = u - v$, the given equations become, respectively,

$$2u^2 + 2v^2 = 6, \text{ and } u^2 - v^2 = 4u - 5; \qquad (3)$$

therefore, eliminating v^2 and simplifying,

$$u^2 - 2u + 1 = 0,$$

whence $u = 1.$

Substituting this value of u in either one of Eqs. (3), gives

$$v = \pm\sqrt{2},$$

whence (since $x = u + v$, and $y = u - v$),

$$x = 1 \pm \sqrt{2}, \text{ and } y = 1 \mp \sqrt{2}.$$

Ex. 6. Solve the equations $\begin{cases} x^3 + y^3 = xy - 5, & (1) \\ x + y + 1 = 0. & (2) \end{cases}$

Solution. On putting $x = u + v$ and $y = u - v$, (1) and (2) become, respectively,

$$2u^3 + 6uv^2 - u^2 + v^2 + 5 = 0, \tag{3}$$

and

$$2u + 1 = 0. \tag{4}$$

From (4) $u = -\frac{1}{2}$,

and substituting this value in (3) gives

$$v = \pm \tfrac{3}{2},$$

whence $x = 1$ or -2, and $y = -2$ or 1.

EXERCISE XCIV

By the method of 134 (i) solve the following systems:

7. $\begin{cases} x^2 + y^2 = 13, \\ xy = 6. \end{cases}$

8. $\begin{cases} u^2 + v^2 = 61, \\ u + v = 11. \end{cases}$

9. $\begin{cases} m + n = 24, \\ \dfrac{mn}{9} = -9. \end{cases}$

10. $\begin{cases} x^2 + y^2 = 1, \\ 25xy + 12 = 0. \end{cases}$

11. $\begin{cases} m = n + \frac{1}{15}, \\ 5mn - 2 = 0. \end{cases}$

12. $\begin{cases} x^2 + y^2 = a, \\ x + y = b. \end{cases}$

By the method of 134 (ii) solve the following systems:

13. $\begin{cases} r^2 - s^2 = 77, \\ r - s = 7. \end{cases}$

14. $\begin{cases} r^2 - s^2 = 77, \\ r + s = 7. \end{cases}$

15. $\begin{cases} r^3 - p^3 = 91, \\ r - p = 7. \end{cases}$

16. $\begin{cases} x^3 + y^3 = a, \\ x + y = b. \end{cases}$

By the method of 134 (iii) solve the following systems:

17. $\begin{cases} x^2 + y^2 = 26, \\ x + y = 6. \end{cases}$

18. $\begin{cases} 3(x - y) = -4xy, \\ x + y = 2. \end{cases}$

19. $\begin{cases} x^2y + xy^2 = -12, \\ x^3 + y^3 = 37. \end{cases}$

20. $\begin{cases} 2(x + y) = -\dfrac{3xy}{5}, \\ x - y = 7. \end{cases}$

21. $\begin{cases} 2x^2 - xy + 2y^2 = 62, \\ x - y - 3xy = -61. \end{cases}$

22. $\begin{cases} s^2 + st + t^2 = 28, \\ s + 2st + t = 22. \end{cases}$

Solve the following systems of equations, choosing for each the method (§§ 131–134) which seems to you best:

23. $\begin{cases} 1+x=y, \\ x^2+y^2=61. \end{cases}$

24. $\begin{cases} \dfrac{x}{y}-\dfrac{y}{x}=\dfrac{16}{15}, \\ x-y=2. \end{cases}$

25. $\begin{cases} x^2-xy=6, \\ x^2+y^2=61. \end{cases}$

26. $\begin{cases} \dfrac{1}{x^2}+\dfrac{1}{y^2}=74, \\ \dfrac{1}{x}-\dfrac{1}{y}=2. \end{cases}$ [cf. Ex. 3, § 104.]

27. $\begin{cases} \dfrac{1}{x^3}+\dfrac{1}{y^3}=91, \\ \dfrac{1}{x}+\dfrac{1}{y}=7. \end{cases}$

28. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{2}, \\ \dfrac{1}{xy}+\dfrac{1}{18}=0. \end{cases}$

29. $\begin{cases} \dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{10}{3}, \\ x^2+y^2=45. \end{cases}$

30. $\begin{cases} \dfrac{1}{u^2}-\dfrac{1}{v^2}=119, \\ \dfrac{1}{u}-7=\dfrac{1}{v}. \end{cases}$

31. $\begin{cases} x^2+y^2+x=y+14, \\ xy=6. \end{cases}$

32. $\begin{cases} m^2n^2=96-4mn, \\ m+n=6. \end{cases}$

33. $\begin{cases} x+y=25, \\ \sqrt{x}+\sqrt{y}=7. \end{cases}$

34. $\begin{cases} x+y+2\sqrt{x+y}=24, \\ x-y+3\sqrt{x-y}=10. \end{cases}$

35. $\begin{cases} 2(x^2+y^2)=5xy, \\ \dfrac{1}{x}+\dfrac{1}{y}=1.5. \end{cases}$

36. $\begin{cases} x^2-2xy=3y^2, \\ y(x+y)=4. \end{cases}$

37. $\begin{cases} (2+x)(y+1)=4, \\ \sqrt{2+x}-\sqrt{y+1}=\frac{1}{6}. \end{cases}$

38. $\begin{cases} a^4+b^4=17, \\ a^2b^2=4. \end{cases}$

39. $\begin{cases} x^2+y^2+6\sqrt{x^2+y^2}=55, \\ x^2-y^2=7. \end{cases}$

40. $\begin{cases} t^2-v^2=165, \\ tv=-26. \end{cases}$

41. $\begin{cases} s^4+s^2y^2+y^4=9, \\ s^2+sy+y^2=3. \end{cases}$

42. $\begin{cases} s^3-t^3=37, \\ st(s-t)=12. \end{cases}$

43. $\begin{cases} x^4+y^4=97, \\ x+y=-1. \end{cases}$

44. $\begin{cases} 3mn+\dfrac{3m}{n}=5, \\ 3mn+\dfrac{3n}{m}=2.5. \end{cases}$

45. $x-\dfrac{b^2}{y}=\dfrac{a^2}{x}-y=a-b.$

PROBLEMS

1. The sum of two numbers is 14, and the difference of their squares is 28. What are the numbers?

2. Find two numbers whose difference is 15, and such that if the greater is diminished by 12, and the smaller increased by 12, the sum of the squares of the results will be 261.

3. Find two numbers whose difference is 80, and the sum of whose square roots is 10.

4. Given that one root of a quadratic equation is 4 times the other and that their product is $\frac{1}{9}$, find the roots; then form the equation which has these roots (cf. § 72).

5. The sum of the roots of a quadratic equation is 12, and their product is -189. What is the equation?

6. The sum of two numbers, their product, and also the difference of their squares are all equal; find the numbers.

7. If the length of the diagonal of a rectangular field, containing 30 acres, is 100 rods, how many rods of fence will be required to inclose the field?

8. Find the dimensions of a rectangular field whose perimeter is 188 rods and whose area will remain unchanged if the length is diminished by 4 rods and the width increased by 2 rods.

9. The sum of the circumferences of two circular flower beds is $56\frac{4}{7}$ feet, and the sum of their areas is $141\frac{3}{7}$ square feet. Find the radius of each. (Cf. Ex. 14, p. 201.)

10. A circular table whose radius is $3\frac{1}{2}$ feet has the same area as a rectangular table whose length is 5 inches more than its breadth. Find the dimensions of the rectangular table.

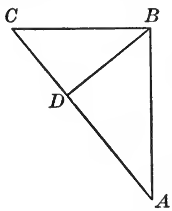
11. A sum of money lent at a certain rate of interest gives an annual income of \$450; if the sum were \$500 more and the rate 1% less, the annual income would be \$50 less. Find the principal and the rate.

12. A sum of money at interest for one year at a certain rate amounted to \$11,130. If the rate had been 1% less and the principal \$100 more, the amount would have been the same. What was the principal, and what the rate?

13. A formal rectangular flower garden is to be enlarged by a border whose uniform width is 10 % of the length of the garden. If the area of the border is 900 sq. ft., and the width of the old garden is 75 % of the width of the new one, find the dimensions of the garden and the width of the border.

14. A certain kind of cloth loses 2 % in width and 5 % in length by shrinking. Find the dimensions of a rectangular piece of this cloth whose shrinkage in perimeter is 38 in., and in area 8.625 sq. ft.

15. The perimeter of a right-angled triangle is 24 ft., and its area is 24 sq. ft. Find the length of each side in the triangle.



16. In the right-angled triangle ABC, BD is drawn perpendicular to AC. If $BC=12$, $AC=17$, and $BD=\sqrt{AD \cdot DC}$, find BD, AD, and DC.

17. The combined capacity of two cubical coal bins is 2728 cu. ft., and the sum of their lengths is 22 ft.; find the length of the diagonal of the smaller bin.

18. Find two numbers whose product is 8 greater than twice their sum, and 48 less than the sum of their squares.

19. Find two numbers such that the sum of their fourth powers is 881 while the sum of their squares is 41.

20. The total area of the walls and ceiling in a room 9 ft. high is 575 sq. ft. Find the length and breadth of the room if their sum is 24 ft.

21. A farmer found that he could buy 16 more sheep than cows for \$100, and that the cost of 3 cows was \$15 greater than the cost of 12 sheep. What was the price of each?

22. If 5 times the sum of the digits of a certain two-digit number is subtracted from the number, its digits will be interchanged; and if the number is multiplied by the sum of its digits, the product will be 648. What is the number?

23. Find two numbers such that the square of either of them equals 112 diminished by 12 times the other.

24. If 5 is added to the numerator and subtracted from the denominator of a given fraction, the result equals the reciprocal of the fraction; and if 2 is subtracted from the numerator, the result equals $\frac{7}{9}$ of the original fraction. Find the fraction.

25. The distance (s) in meters, through which a body falling from a position of rest passes in the tth second of its fall is given by the formula $s = \frac{1}{2}g(2t - 1)$; and the total distance (S) fallen in t seconds is $S = \frac{1}{2}gt^2$. How long has a body been falling when $s = 44.1$ meters and $S = 122.5$ meters? If g is less than 10, what is its value?

26. Solve the problem of Ex. 25 if s and S are each expressed in feet, and $s = 112\frac{7}{12}$ and $S = 257\frac{1}{3}$.

27. In going 40 yards more than $\frac{1}{4}$ of a mile the fore wheel of a carriage revolves 24 times more than the hind wheel; but if the circumference of each wheel were 3 ft. greater, the fore wheel would revolve 16 times more than the hind wheel. What is the circumference of the hind wheel?

28. A merchant paid \$125 for an invoice of two grades of sugar. By selling the first grade for \$91, and the second for \$36, he gained as many per cent on the first grade as he lost on the second. How much did he pay for each grade?

29. Two trains start at the same time from stations A and B, 320 miles apart, and travel toward each other. If it requires 6 hr. and 40 min., from the time the trains meet, for the first train to reach B, and 2 hr. and 24 min. for the second to reach A, find the rate at which each train runs.

30. After traveling 2 hr., a train is detained 1 hr. by an accident; it then proceeds at 60 % of its former rate, and arrives 7 hr. 40 min. late. Had the accident occurred 50 miles farther on, the train would have been 6 hr. 20 min. late. Find the distance traveled by the train. (Cf. Ex. 42, p. 164.)

135.* **Simultaneous quadratics not always solvable by methods already given.** While many systems containing quadratics (and

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* § 135 may, if the teacher prefers, be omitted till the subject is reviewed.

some containing still higher equations) may be solved by the methods of §§ 131–134, these methods do not always suffice for the solution of such systems.

Thus, inspection shows that the system

$$\begin{cases} x^2 - 3x + 8y = 4, \\ 3x^2 - 16y^2 + 20y = 9, \end{cases}$$

cannot be solved by the methods of §§ 132–134; and elimination by substitution (as in § 131) leads to an equation of the fourth degree in one unknown number, viz., to

$$x^4 - 6x^3 - x^2 - 6x + 12 = 0,$$

which cannot be solved by the elementary methods already studied. Such equations are discussed in higher algebra.

For all systems like the above, however, *approximate* solutions may be obtained by means of graphs (cf. § 143).

136.* **Systems containing three or more unknown numbers.** Some systems containing three or more simultaneous equations, some of which are quadratic, may be solved by elementary methods.

E.g., if one equation of a given system is quadratic, and all the others are of the first degree, then a slight modification of the method of § 131 will provide a solution (cf. *El. Alg.* § 180).

The solution of such systems in general is, however, beyond the limits of this book.

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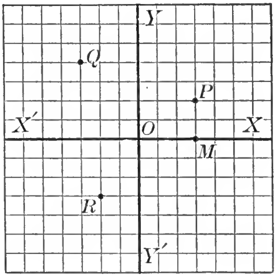
* This article may, if the teacher prefers, be omitted till the subject is reviewed.

CHAPTER XIII

GRAPHIC REPRESENTATION OF EQUATIONS*

137. Introductory. Although an equation in two unknown numbers has (§ 99) an infinitely large number of solutions, and is in that sense indeterminate, yet by a beautiful device, due to the celebrated mathematician and philosopher Descartes (pronounced dā-kärt′, born 1596, died 1650), a *perfectly definite picture* of such an equation may be made (cf. § 139).

138. Axes. Coördinates. Let us draw (as Descartes did) two perpendicular straight lines $X'X$ and $Y'Y$, cutting each other in the point O, and call these lines the **coördinate axes.** If we now agree that distances measured toward the right from $Y'Y$, or upward from $X'X$, shall be positive, while distances toward the left, or downward, shall be negative, then any point in the plane of this page can be located as soon as we know its distances from the axes $X'X$ and $Y'Y$.



Thus, to locate the point P, 3 inches from $Y'Y$ and 2 inches from $X'X$, we measure off 3 inches (represented in the diagram by 3 spaces) toward the right from O, to the point M, say, and then 2 inches upward from M.

The numbers which serve to locate a point (in this case

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*This chapter should be included in the course whenever possible; its omission, however, will not break the continuity of the work.

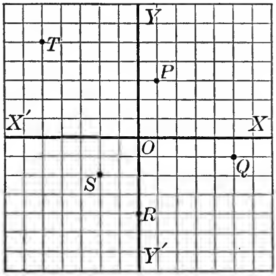
3 and 2) are called the **coördinates** of the point. The point P may be represented by the symbol (3, 2).

Similarly the point Q (– 3, 4) is located by measuring 3 spaces toward the *left* from O, and then four spaces upward. The point R (– 2, – 3), also, is represented in the figure.

NOTE. This plan of locating points in the figure somewhat resembles that used to locate places on the earth's surface by their latitude and longitude. The coördinate axes correspond to the equator and the prime meridian.

EXERCISE XCV

1. Name the x-coördinate (*i.e.*, the distance from the axis $Y'Y$) of each point located in the figure below. Also name the y-coördinates of these points.



Draw a pair of axes as in § 138 and locate the following points:

2. (5, 4); (3, 7); (4, –2); (–3, 1); and (–4, –6).

3. $(-\frac{1}{2}, 1)$; $(\frac{1}{2}, \frac{2}{3})$; $(1\frac{1}{2}, -3)$; $(4, \frac{1}{4})$; and $(-\frac{3}{4}, -5)$.

4. (3, 0); (–5, 0); (0, 8); (0, 0); and (0, –2).

5. Where are the points whose y-coördinate is 0? Where are those whose x-coördinate is 0? those whose y-coördinate is $3\frac{1}{2}$?

6. Locate five points each of which has its x-coördinate equal to its y-coördinate, and draw a line through these points. Does this line contain any other points whose two coördinates are equal?

7. Where are the points which have their y-coördinates opposite in value to their respective x-coördinates?

8. Verify that the equation $2x - y = 3$ is satisfied by each of the following number-pairs: (0, –3); (1, –1); (2, 1); (3, 3); (4, 5); then locate the point corresponding to each pair. On what kind of a line do these points lie?

9. Measure the coördinates of several other points on the line mentioned in Ex. 8. Are these coördinates solutions of $2x - y = 3$?

10. Locate the following points: $(0, 5)$; $(0, -5)$; $(5, 0)$; $(-5, 0)$; $(4, 3)$; $(-4, 3)$; $(4, -3)$; $(-4, -3)$; $(3, 4)$; $(-3, 4)$; $(3, -4)$; $(-3, -4)$. On what kind of a line do they lie?

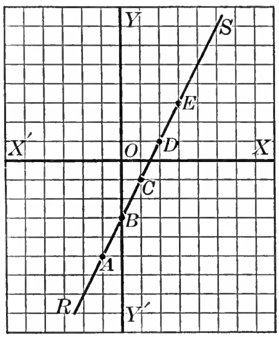
139. The picture (graph) of an equation. Consider the equation

$$2x - y = 3.$$

This equation is, manifestly, satisfied by the following pairs of values of x and y (§ 99):

$(-1, -5)$; $(0, -3)$; $(1, -1)$; $(2, 1)$; $(3, 3)$; $(4, 5)$; etc.

If we now locate (as in § 138) the points A, B, C, etc., corresponding to these number-pairs, we find that they are not scattered at random over the page, but that *they all lie upon the straight line RS* in the figure (cf. Ex. 8, p. 220). Moreover, the coördinates of every point in the line RS, and those of no other points whatever, satisfy the given equation.*



For these reasons, the line RS may be regarded as the *picture* of the equation; it is usually called the graph, also the locus of the equation. That is, the **graph** (or **locus**) of an equation is the line (or lines) containing all the points (and no others) whose coördinates satisfy the given equation.

140. Drawing of graphs. The method illustrated in § 139, for finding the graph of an equation in x and y, may be stated thus:

(1) Solve the given equation for y, in terms of x.

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* Let the pupil test this statement by careful measurement on a large and well-drawn figure. The proof of its correctness follows easily from the theory of similar triangles in geometry.

(2) Assign to x a succession of values, such as 0, 1, 2, 3, $\cdots$ (also -1, -2, -3, $\cdots$), and find the *corresponding* values of y; *i.e.*, find a succession of solutions of the given equation.

(3) By means of a pair of axes locate the points corresponding to these solutions, — use cross-section paper.

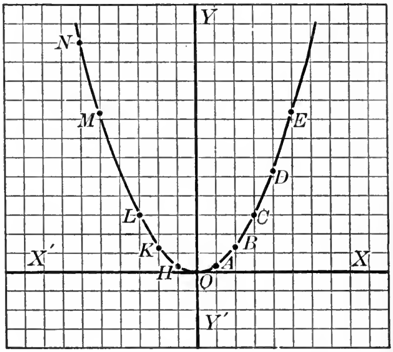
(4) Draw a line connecting these points in regular order; this line is (approximately) the graph of the given equation.

E.g., to find the graph of the equation $3y - x^2 = 0$, we solve the equation for y in terms of x, and tabulate the corresponding values of x and y, thus:

$$y = \tfrac{1}{3}x^2.$$

x	y	Points
0	0	O
1	$\frac{1}{3}$	A
2	$\frac{4}{3}$	B
3	3	C
.	.	.
.	.	.
.	.	.
-1	$\frac{1}{3}$	H
-2	$\frac{4}{3}$	K
-3	3	L
.	.	.
.	.	.
.	.	.

Locating the points O, A, B, C, etc., and connecting them in order, we obtain the line $NML \cdots E$, which



is a good approximation to the graph of the given equation.

By assigning to x values *between* 0 and 1, 1 and 2, etc., and finding the corresponding values of y, we can locate points *between* O and A, A and B, etc., and thus draw a closer approximation to the required graph.

Note. The graph of a *first degree* equation in x and y is (cf. § 139) a *straight line* (hence a first degree equation is often called a *linear equation*). In this case, of course, only two points (*i.e.*, two solutions of the equation) need be found in order to draw the complete graph.

EXERCISE XCVI

1. Find six solutions of $2x + y = 12$, locate the points determined by these solutions, and draw the graph of the equation.

Using the plan of Ex. 1, draw the graph of:

2. $x + 2y = 8$.

3. $x - 2y = 1$.

4. $3x = y$.

5. $2x - 3y = 0$.

6. $3x + 2y = 12$.

7. $2y - x^2 = 0$.

8. Draw the graph of $3x = 2$ [*i.e.*, $3x + 0 \cdot y = 2$, cf. Ex. 5, p. 220]; of $2y = 5$; of $x = -1$; of $x^2 = 9$.

9. What is the graph of $x = 0$? of $y = 0$? Without making a drawing, show that the graph of $2x - 7y = 0$ must pass through the point O in which the axes intersect. Is this true for the graph of *every* equation of the form $ax + by = 0$?

10. In the equation $4x - 5y = 10$, when $x = 0$, $y =$? When $y = 0$, $x =$? From these two solutions of the equation draw its graph (cf. § 140, Note).

Draw the graph of:

11. $x - y = 0$.

12. $x + y = 0$.

13. $3x = -11$.

14. $y^2 - 16 = 0$.

15. $2x + 3y = 6$.

16. $2x - 3y = 6$.

17. $5x - 2y = 20$.

18. $3x + 5y = 7\frac{1}{2}$.

19. $7x - y = 3\frac{1}{2}$.

20. $5x + 2y = 21$.

21. $4x = y^2$.

22. $3x^2 - 4y = 0$.

Calling the coördinate axes $S'S$ and $T'T$ instead of $X'X$ and $Y'Y$, draw the graph of:

23. $4t = 3s$. **24.** $s - t = 5$. **25.** $2t - 3s^2 = 0$.

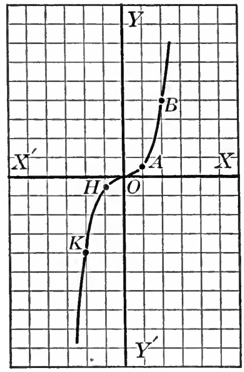
141. Drawing of graphs (continued). Thus far we have considered only the simplest kind of graphs; the method employed will serve, however, for any equations whatever in two unknown numbers.

Ex. 1. Construct the graph of $2y - x^3 = 0$.

Construction. Solving this equation for y in terms of x, and tabulating the corresponding values of x and y, we obtain:

$$y = \tfrac{1}{2}x^3.$$

x	y	Points
0	0	O
1	$\frac{1}{2}$	A
2	4	B
.	.	.
.	.	.
.	.	.
-1	$-\frac{1}{2}$	H
-2	-4	K
.	.	.
.	.	.
.	.	.



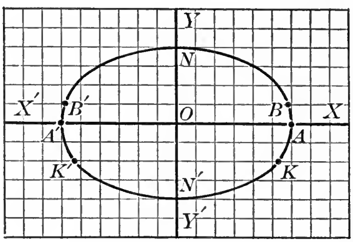
On locating these points and connecting them in order, we obtain the required graph, viz.,: $KHOA \cdots$.

Ex. 2. Construct the graph of $4x^2 + 9y^2 = 144$.

Construction. Proceeding as in Ex. 1, we obtain:

$$x = \tfrac{3}{2}\sqrt{16 - y^2}.$$

y	x	Points
0	6 or -6	A or A'
1	$\frac{3}{2}\sqrt{15}$ or $-\frac{3}{2}\sqrt{15}$	B or B'
.	. .	. .
.	. .	. .
.	. .	. .
-1	$\frac{3}{2}\sqrt{15}$ or $-\frac{3}{2}\sqrt{15}$	H or H'
-2	$3\sqrt{3}$ or $-3\sqrt{3}$	K or K'
.	. .	. .
.	. .	. .
.	. .	. .



On locating these points, using approximate values of the square roots, and connecting them by a smooth curve, we obtain the graph $ABNA'N'A$.

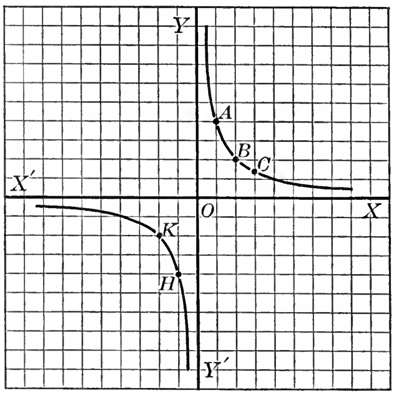
NOTE. The limitations of the graph in Ex. 2 are interesting. Thus, since $x = \frac{3}{2}\sqrt{16 - y^2}$, x must be imaginary when y is greater than 4; hence, as our graphic representation admits *real* values only, there are no points on the curve whose y-coördinate is greater than 4. Similarly, it may be shown that there are no points on the graph below $y = -4$. And solving the given equation for y in terms of x shows that there are no points on the graph at the right of $x = 6$, or at the left of $x = -6$.

Ex. 3. Construct the graph of $xy = 4$.

CONSTRUCTION. Proceeding as in Exs. 1 and 2, we obtain:

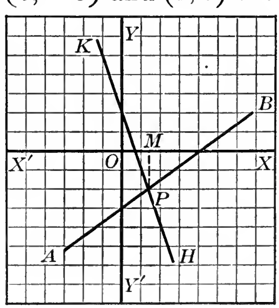
$$y = \frac{4}{x}.$$

x	y	POINTS
0	∞	
1	4	A
2	2	B
3	$\frac{4}{3}$	C
.	.	.
.	.	.
.	.	.
-1	-4	H
-2	-2	K
.	.	.
.	.	.
.	.	.



On locating these points and connecting them by a smooth curve, we obtain the graph $AB \cdots HK$.

142. Intersection of graphs. Since $(0, -3)$ and $(4, 0)$ are solutions of the equation $3x - 4y = 12$, therefore its graph is the line AB in the figure (cf. § 140, Note).



If we now draw the graph of $3x + y = 2$, using the same axes as before, we obtain the line HK.

Moreover, since P, the point in which AB and HK intersect (*i.e.* cut) each other, lies on each of these

graphs, therefore its coördinates (§ 138) must satisfy each of the given equations (cf. § 139).

Hence, we may find the coördinates of P by merely solving the given equations as in § 101, and without even drawing their graphs.

Approximate values of the coördinates of P may, of course, be found by direct measurement of OM and MP; this measurement constitutes a *graphical solution* of the given equations. Let pupils use both methods for finding these coördinates, and compare results.

REMARK. From what has just been said, and from the definitions in § 100, it follows that (let pupils explain why):

(1) The graphs of consistent equations intersect each other.

(2) The graphs of inconsistent linear equations are parallel lines.

EXERCISE XCVII

Construct the graph of:

4. $y^2 = 8x$.

5. $y = (x-1)^2$.

6. $x^2 + y^2 = 25$.

7. $16x^2 + y^2 = 64$.

8. $xy = 5$.

9. $3x^2 + 4y^2 = 12$.

10. $3x^2 - 4y^2 = 12$.

11. $4y^2 = x^2$.

12. Show from the equation of Ex. 4 that no part of its graph lies to the left of the y-axis (the line $Y'Y$).

13. Show from the equation of Ex. 6 that no part of its graph lies outside a certain square whose side is 5; similarly, show that the graph of Ex. 7 is contained within a certain rectangle whose dimensions are 16 and 4.

14. Show from the equation of Ex. 8 that its graph consists of two infinitely long branches, one in the quarter XOY and one in the quarter $X'OY'$.

15. If the graph of $xy = -5$ were drawn, how would it differ from that of $xy = 5$? Why?

16. Draw the graph of $2x+y=x^2+3$ and show that this graph differs from that of Ex. 5 only in being moved two divisions upward. Explain why this should be so.

17. Find, both by solving the equations, and by measurement, the coördinates of the point in which the graphs of $x+y=5$ and $2x-y=4$ intersect; compare your results.

In each of Exs. 18–23 below find, as in Ex. 17, the coördinates of the point in which the graphs of the two equations intersect:

18. $\begin{cases} x=-2, \\ 2y-x=6. \end{cases}$

19. $\begin{cases} 2x=7y, \\ y-x=5. \end{cases}$

20. $\begin{cases} 3y+2x=17, \\ 2x-y=5. \end{cases}$

21. $\begin{cases} x+y=3, \\ 2x+2y=8. \end{cases}$

22. $\begin{cases} x+y=3, \\ \frac{1}{2}x+\frac{1}{2}y=1\frac{1}{2}. \end{cases}$

23. $\begin{cases} 2x-y=3, \\ 3y-x^2=0. \end{cases}$ (cf. §§ 139, 140.)

24. How are the graphs of two first degree equations in x and y related when the equations are inconsistent (cf. Ex. 21)? when they are simultaneous and independent (cf. Ex. 20)? simultaneous and *not* independent (cf. Ex. 22)?

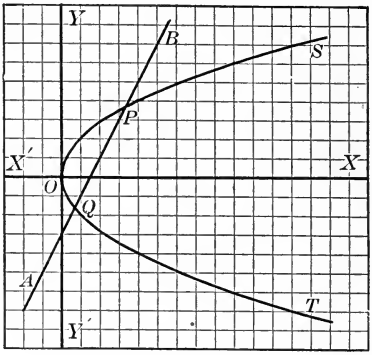
143. Graphic solution of simultaneous equations. If the graph of one of two simultaneous equations is drawn across the graph of the other (*i.e.*, if the same axes are used in both drawings), then the measured coördinates of each point in which these graphs intersect constitute an approximate solution of the given system (cf. § 142). The following examples will illustrate this procedure.

Ex. 1. Solve graphically the simultaneous equations

$$\begin{cases} 3x-4y=12, \\ 3x+y=2. \end{cases}$$

Solution. The graphs of these equations are the lines AB and HK (figure, § 142); and the (measured) coördinates of P, their point of intersection, are approximately $x=\frac{4}{3}$ and $y=-2$, which, by trial, are found to be a solution of the given system.

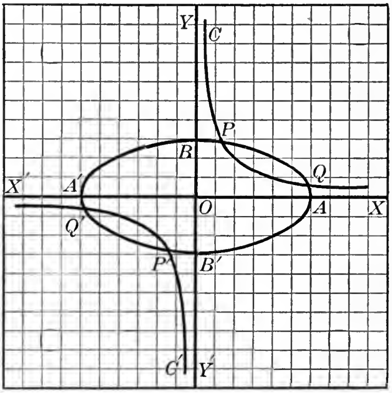
Ex. 2. Solve graphically the system $\begin{cases} y^2 = 4x, \\ y + 3 = 2x. \end{cases}$



SOLUTION. The graphs of these equations are, respectively, $SPOQT$ and AB. The coördinates of P, one of their points of intersection, are approximately $x = 3.4$ and $y = 3.7$, which constitute an approximate solution of the given equations.

So, too, the cöordinates of Q (viz., $x = .65$ and $y = -1.6$) constitute an approximate solution of the system.

Ex. 3. Solve graphically the system $\begin{cases} x^2 + 4y^2 = 36, \\ xy = 4. \end{cases}$



SOLUTION. The graphs of these equations are, respectively, $AB'A'B$ and $CPQC'P'Q'$; and the coördinates of P, one of their points of intersection, are approximately $x = 1.4$ and $y = 2.9$, which constitute an approximate solution of these equations.

So, too, the coördinates of Q ($x = 5.75$ and $y = .7$), P'($x = -1.4$ and $y = -2.9$), and Q' ($x = -5.75$ and $y = -.7$) are approximate solutions of the given equations.

NOTE TO THE TEACHER. In the case of a pair of simple equations the solution by the method of § 101 is usually easier than the graphic method, and its results are *exact* instead of *approximate*. There are, however, many other cases in which the graphic method is advantageous; hence some practice with it, even on simple equations, is recommended.

EXERCISE XCVIII

Solve graphically the following systems of equations, and check your results as the teacher directs:

4. $\begin{cases} x+y=3, \\ x-y=3. \end{cases}$

5. $\begin{cases} 2x-y=5, \\ 4x=16-y. \end{cases}$

6. $\begin{cases} 4y+3x=5, \\ x=5. \end{cases}$

7. $\begin{cases} x+y=4, \\ y=2-x. \end{cases}$

8. $\begin{cases} 5x-10y=36, \\ 2x+3y=-8. \end{cases}$

9. $\begin{cases} 4x+\frac{2}{3}y=6, \\ \frac{5}{9}x-\frac{3}{2}y=8. \end{cases}$

10. $\begin{cases} 2x=y^2, \\ 2y=x. \end{cases}$

11. $\begin{cases} 2x-y^2-1=0, \\ 2x+6y+7=0. \end{cases}$

12. $\begin{cases} x-2y=-12, \\ y-x^2=-2x-2. \end{cases}$

13. $\begin{cases} x^2+y^2=25, \\ y=3. \end{cases}$

14. $\begin{cases} x^2+y^2=25, \\ x+y=1. \end{cases}$

15. $\begin{cases} y=3x+2, \\ x^2=4-y^2. \end{cases}$

16. $\begin{cases} y^2=\dfrac{4x}{2-x}, \\ x=1. \end{cases}$

17. $\begin{cases} xy=-10, \\ x+y=2. \end{cases}$

18. $\begin{cases} x^2+9=y, \\ y=x^2-5x+6. \end{cases}$

19. $\begin{cases} x^2+y^2=25, \\ xy=-4\frac{1}{2}. \end{cases}$

20. $\begin{cases} 4x^2-9y^2=36, \\ x^2+y^2=25. \end{cases}$

21. $\begin{cases} y=5x-15, \\ x^3-9x^2+23x-15=y. \end{cases}$

By referring to the graphs in the above exercises, find the number of solutions of a system consisting of:

22. Two simple equations.

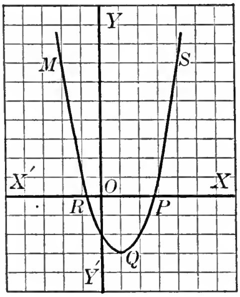
23. A simple and a quadratic equation.

24. Two quadratic equations.

144. Graphic solution of equations containing but one unknown number. By slightly extending the method of § 143, we may find graphic solutions for quadratic equations in one unknown number.

Thus, the roots of $x^2-2x-2=0$ are manifestly the values of x found by solving the pair of simultaneous equations:

$$\begin{cases} y=x^2-2x-2, & (1) \\ y=0. & (2) \end{cases}$$



Now the graph of (2), viz., the x-axis, cuts the graph of (1), viz., the curve MQS, in the points P and R, whose coördinates are (approximately) $x=2.75$, $y=0$, and $x=-.75$, $y=0$. And since (§ 143) each of these pairs of values constitutes an approximate solution of (1) and (2), therefore 2.75 and $-.75$ are approximate roots of $x^2-2x-2=0$.

EXERCISE XCIX

Find graphic solutions of:

1. $x^2-6x+8=0$.

2. $x^2-3x+5=0$.

3. $6x^2+5x-4=0$.

4. $x^3-3x^2-6x+8=0$.

5. Show graphically that $4x^3-4x^2-11x+6=0$ has one root between 0 and 1, and a second root between -1 and -2. What is the third root of this equation?

6. Show that one root of $x^3-7x^2+9x=1$ lies between 1 and 2. Between what integers do each of the other two roots lie?

7. Corresponding to any given value of x, how does the value of y in $y=x^2-6x+6$ compare with its value in $y=x^2-6x+7$? Could, then, the graph of the second equation be obtained by merely moving that of the first upward through one division?

8. Compare the graphs of $y=2x^2-10x-3$ and $y=2x^2-10x+1$; also those of $y=3+4x-x^2$ and $y=10+4x-x^2$.

9. By first constructing the graphs of $y=x^2-6x+6$, $y=x^2-6x+7$, etc., compare the roots of $x^2-6x+6=0$, $x^2-6x+7=0$, $x^2-6x+8=0$, $x^2-6x+9=0$, $x^2-6x+10=0$, and $x^2-6x+11=0$.

10. As in Ex. 9 compare the two smaller roots of $x^3-7x^2+9x-1=0$ with those of $x^3-7x^2+9x-3=0$ and $x^3-7x^2+9x-5=0$.

NOTE. Exercises 9 and 10 illustrate how, by changing the absolute term in an equation, a pair of unequal roots can be made gradually to become equal and then imaginary.

11. Show that the roots of $x^2-2x-2=0$ (§ 144) can be found from the graphic solution of the system

$$\begin{cases} y=x^2, & (1) \\ y-2x-2=0. & (2) \end{cases}$$

12. Show that the graph of Eq. (1) in Ex. 11 may be used in the solution of other quadratic equations (*e.g.*, $x^2+5x=7$) also.

13. Is the method given at the top of p. 230, or that suggested in Exs. 11 and 12, to be preferred when we have *several* quadratic equations to solve graphically? Explain.

By the method of Ex. 11 solve:

14. $x^2-2x-2=0$.

15. $x^2-x=0$.

16. $x^2+x-4=0$.

17. $x^2=x+3$.

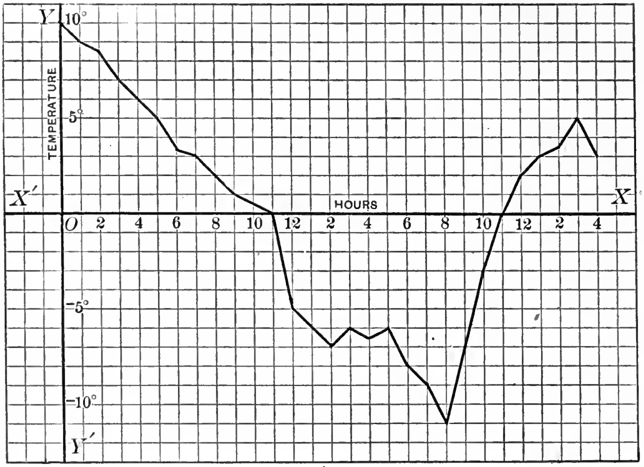
18. $12x-4x^2=5$.

19. $2x^2-x=-3$.

145. Use of graphs in physics, engineering, statistics, etc. Descartes's plan for graphically representing equations has now been adopted by practically all scientific men to represent simultaneous changes in related quantities. Physicists, chemists, engineers, physicians, statisticians, etc., all find that this graphic representation of related changes often gives at a glance information which could be secured otherwise only by considerable effort, and that it often brings out facts of importance which might otherwise escape notice.

As a simple example of the use of graphs in this way let us consider the following temperature readings, taken from the U.S. Weather Bureau report, for 28 hours beginning at noon on Feb. 5, 1906, at Ithaca, N.Y.

HR.	TEM.
12	10°
1	9
2	8.5
3	7
4	6
5	5
6	3.5
7	3
8	2
9	1
10	0.5
11	0
12	−5
1	−6
2	−7
3	−6
4	−6.5
5	−6
6	−8
7	−9
8	−11
9	−7
10	−3
11	0
12	2
1	3
2	3.5
3	5
4	3



Here the tabulated figures and the graph answer the same questions as to the temperature — when, how much, and how rapidly it rose or fell, what were its highest and lowest points, etc. The graph, however, yields this information at a mere glance, while the tabulated figures yield it only after they have been studied and compared.

NOTE. For other interesting applications of graphs see Tanner and Allen's *Analytic Geometry*, pp. 73–78. Also, and especially, "Graphic Methods in Elementary Algebra," by Prof. William Betz, in *School Science and Mathematics*, vol. 6, pp. 683–687. This article gives many good suggestions as well as valuable material and references.

EXERCISE C

By reference to the above temperature *graph* (usually called *thermograph*), answer the following questions:

1. Between what hours was the temperature below 0°? When was it lowest?

2. When was the temperature falling most rapidly? Explain.

The following tables give the population (in millions) of the countries named, for certain years between 1800 and 1900.

British Isles		Lands now in the German Empire		France		United States	
Year	Population	Year	Population	Year	Population	Year	Population
	(millions)		(millions)		(millions)		(millions)
1801	15.9	1816	24.8	1801	27.3	1810	7.2
1811	17.9	1837	31.5	1821	30.4	1820	9.6
1821	20.9	1847	34.7	1841	34.2	1830	12.8
1831	24	1856	36.1	1861	37.3	1840	17
1841	26.7	1865	39.4	1866	38	1850	23.2
1851	27.3	1872	41	1872	36.1	1860	31.4
1861	28.9	1876	42.7	1876	36.9	1870	38.5
1871	31.4	1885	46.8	1881	37.6	1880	50.1
1881	34.9	1895	52.2	1891	38.3	1890	62.6
1891	37.7			1896	38.5		

3. Taking the number of years after 1800 as the x-coördinate and the population (in millions) as the y-coördinate, locate the several points represented by the above table for the British Isles, and join these points by straight lines. Similarly, draw graphs for the remaining tables.

4. By reference to your *graphs*, compare the population of the countries named in 1830; in 1856; in 1880. By reference to the *tables* compare the populations in 1871.

5. In making comparisons like those of Ex. 4, is it easier to use the tables or to use the graphs? Why (cf. § 145)?

6. By reference to the graphs answer the following questions:

(1) When did the population of the United States first exceed that of the British Isles? that of France? that of Germany?

(2) During what years has the population of the United States increased most rapidly?

CHAPTER XIV

IRRATIONAL NUMBERS — RADICALS

146. Preliminary remarks and definitions. A number which may be expressed as the quotient of two integers, positive or negative, is called a **rational number.**

E.g., $3(=\frac{3}{1})$; $-7\frac{1}{2}(=\frac{-15}{2})$; $2.75(=\frac{275}{100})$, etc., are rational numbers.

Nearly all the numbers thus far used have been rational, although we have met a few such forms as $\sqrt{2}$ and $\sqrt{-5}$.

In this and the next chapter we shall examine more closely such numbers as $\sqrt{2}$ and $\sqrt{-5}$. These numbers are particular cases of $\sqrt[n]{a}$, which is defined (§ 113) by the equation $(\sqrt[n]{a})^n = a$; hence $(\sqrt{2})^2 = 2$ and $(\sqrt{-5})^2 = -5$.

The numbers $\sqrt{2}$ and $\sqrt{-5}$ resemble each other in that neither of them is rational (since no rational number squared is 2 or -5), but, as we shall soon see, they differ widely in another regard.

By squaring 1 and 2, we find that $\sqrt{2}$ is greater than 1 and less than 2; then by squaring 1.1, 1.2, 1.3, ⋯, we find that $\sqrt{2}$ is greater than 1.4 and less than 1.5; similarly, $\sqrt{2}$ is greater than 1.41 and less than 1.42; etc.

Since $\sqrt{2}$ lies *between* 1 and 2, 1.4 and 1.5, 1.41 and 1.42, etc., therefore, if 1.4 (or 1.5) is taken for $\sqrt{2}$, the error is less than 0.1; if 1.41 (or 1.42) is taken, the error is less than 0.01; and, by continuing this process, we can find rational numbers which approximate $\sqrt{2}$ to any required degree of accuracy.

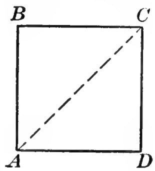
On the other hand, since the square of any rational number is *positive*, therefore we cannot express $\sqrt{-5}$, even approximately, by means of rational numbers.

Numbers like $\sqrt{2}$, which are not rational, but which may be expressed approximately to any required degree of accuracy by means of rational numbers, are called **irrational numbers.**

E.g., $\sqrt{2}$, $5-\sqrt[3]{7}$, and $10+\sqrt[3]{2}$ are irrational numbers.

Numbers like $\sqrt{-5}$, which cannot be expressed, even approximately, by means of rational numbers, are called **imaginary numbers** (cf. § 114, Note). Rational and irrational numbers taken together are called **real numbers.**

NOTE TO THE TEACHER. Emphasis should be laid upon the fact that although such numbers as $\sqrt{2}$ can be expressed only approximately by means of rational numbers, they are, nevertheless, just as *exact* and *definite* as are integers and fractions.



Thus, let $ABCD$ be a square whose side AB is 1 foot long, and let x represent the number of feet in its diagonal AC, then it is easily proved by geometry that

$$x^2 = 2, \text{ i.e., that } x = \sqrt{2}.$$

The numbers 1, 1.4, 1.41, 1.414, 1.4142, etc., are successive *approximations* to the length of this diagonal, but its *exact* length is $\sqrt{2}$; hence the necessity of including such numbers as $\sqrt{2}$ in our number system.

It will be worth while also to connect this latest extension with the extensions previously made (see p. 16, footnote). Thus fractions arose from generalizing division; negative numbers arose from generalizing subtraction; and in the present article it appears that generalizing evolution introduces two further new kinds of numbers, viz., the irrational and the imaginary.

In other words: while the *direct* operations (viz., addition, multiplication, and involution) with positive integers always produce results that are positive integers, the *inverse* operations (viz., subtraction, division, and evolution) lead respectively to negative, fractional, and irrational and imaginary numbers, and demand for their accommodation that the primitive idea of number be so enlarged as to include these new kinds of numbers.

147. Further definitions. An indicated root is usually called a **radical**; the number whose root is indicated is called the **radicand**. If the root is irrational, but the radicand rational, the expression is often called a **surd**. Thus,

$\sqrt{2}$, $\sqrt[3]{8}$, $6\sqrt[3]{45}$, $\sqrt{-2}$, and $\sqrt[7]{5+\sqrt{10}}$ are radicals, whose respective radicands are 2, 8, etc.; of these radicals $\sqrt{2}$ and $6\sqrt[3]{45}$ alone are called surds.

The **coefficient** of a radical is the factor which multiplies it, and the **order** of the radical is determined by the root index. Two radicals which have the same root index are said to be of the **same order**. Thus, the surds $12\sqrt[7]{5\,ax^2}$ and $m^2\sqrt[7]{674}$ are of the same order, viz., the 7th, and their coefficients are 12 and m^2, respectively.

Surds of the second and third orders are usually called **quadratic** and **cubic** surds, respectively.

Radicals which, when simplified, are of the same order and have their radicands exactly alike are called **similar** (also **like**) radicals; otherwise they are **dissimilar (unlike)**. Expressions which involve radicals, in any way whatever, are called **radical expressions**; they are **monomial, binomial,** etc. (cf. § 20), depending upon the number of their terms. Thus, $\sqrt{5}$ and $3\sqrt{5}$ are similar, monomial, quadratic surds, while $5\,a+3\sqrt{7}$ and $2\sqrt[5]{9}+3\sqrt{x}$ are binomial surds.

148. Principal roots. We have already seen that a number has *two* square roots (*e.g.*, $\sqrt{9}$ is $+3$ or -3), and we shall see later that every number has *three* cube roots, *four* fourth roots, *five* fifth roots, etc.

E.g., $\sqrt[3]{8}=2$, $-1+\sqrt{-3}$, or $-1-\sqrt{-3}$, since the cube of each of these numbers is 8; and $\sqrt[4]{16}=2$, -2, $2\sqrt{-1}$, or $-2\sqrt{-1}$.

Although the number of roots always equals the order of the radical, not more than two of these roots can be *real;* and when there are two real roots, they are opposite numbers. By the **principal root** of a number is meant its *real root*, if there is but *one* real root, and its *real positive root* if there are *two* real roots.

E.g., if attention is confined to principal roots, $\sqrt{9}=3$ (and not -3), $\sqrt[3]{-8}=-2$, $\sqrt[3]{125}=5$, $\sqrt[4]{16}=2$, etc.

EXERCISE CI

1. What is a rational number? Use your answer to show that 7, $\frac{3}{5}$, $-8\frac{1}{4}$, and $\sqrt{36}$ are all rational.

2. What is an irrational number? Is $\sqrt[3]{8}$ an irrational number? Why?

3. By the method used in § 146 for $\sqrt{2}$, find two approximate values for $\sqrt{3}$ (one larger and the other smaller than the true value) which differ from $\sqrt{3}$ by less than 0.001.

4. Find two successive approximations to the value of $\sqrt{5}$. Compare these approximations with the result of extracting the square root of 5 by the method of § 118.

5. What is an imaginary number? Give several illustrations. For what values of n is $\sqrt[n]{-5}$ imaginary?

6. Is the number $21+\sqrt{17}$ rational or irrational? Why? What kind of number is $84\sqrt{5}-\sqrt[4]{-8}$? Why?

7. Are both $\sqrt[3]{21}$ and $\sqrt[3]{2+\sqrt{7}}$ radicals? Are they surds? Are all radicals surds? Are all surds radicals?

8. In Exs. 44–54, p. 242, point out the coefficient of each surd. May the coefficient of a surd be fractional? negative?

9. Write a surd of each of the following orders: 2d, 5th, 3d, 7th.

10. Define similar surds, and illustrate your definition. May the coefficients differ and the surds still be similar?

11. What factor do two similar surds necessarily have in common? What kind of a number, then, is the quotient of two similar surds? Illustrate your answer.

12. Write a monomial cubic surd; a binomial quadratic surd; a trinomial surd of the 5th order.

13. How many values has $\sqrt{16}$? What are they? What is the principal square root of 16? What is the principal fifth root of -32? Define the principal root of a number.

149. Principles involved in operations with radicals. If we exclude imaginary numbers, the principles employed in operations with radicals may be symbolically stated thus:

$$\text{(i)} \qquad \sqrt[n]{xy} = \sqrt[n]{x} \cdot \sqrt[n]{y},$$

$$\text{(ii)} \qquad \sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}},$$

$$\text{(iii)} \qquad \sqrt[nt]{x} = \sqrt[n]{\sqrt[t]{x}} = \sqrt[t]{\sqrt[n]{x}};$$

wherein n and t are positive integers, while x and y may have any values whatever, except that they cannot be negative when the root index is even.

150. Proof of the principles in § 149. For the sake of simplicity, we shall (1) limit the proofs to principal roots, and (2) assume that a change in the *order* of the factors of a product, even when these factors are irrational numbers, leaves the product unchanged (cf. § 42).

With these restrictions the correctness of (i), (ii), and (iii), § 149, follows from the meaning of the symbol $\sqrt[n]{a}$ (§ 113). Thus, to prove (i) we proceed as follows:

$$(\sqrt[n]{x}\sqrt[n]{y})^n = \sqrt[n]{x}\sqrt[n]{y} \cdot \sqrt[n]{x}\sqrt[n]{y} \cdot \cdots \text{ to } n \text{ factors} \qquad [\text{§ } 9$$

$$= (\sqrt[n]{x})^n \cdot (\sqrt[n]{y})^n$$

$$= xy; \qquad [\text{since } (\sqrt[n]{a})^n = a$$

whence $\sqrt[n]{x}\sqrt[n]{y} = \sqrt[n]{xy}$, [§ 113

which was to be proved.

This principle may be translated into words thus:

The nth root of the product of two numbers equals the product of the nth roots of these numbers.

The proofs of (ii) and (iii) are left as an exercise for the pupil.

EXERCISE CII

Verify the following equations:

1. $\sqrt{9}\cdot\sqrt{25}=\sqrt{9\cdot 25}$. **3.** $\sqrt{16\cdot 9}=\sqrt{16}\cdot\sqrt{9}$.

2. $\sqrt[3]{-8}\cdot\sqrt[3]{27}=\sqrt[3]{-8\cdot 27}$. **4.** $\sqrt[3]{1000\,a^6}=\sqrt[3]{125\,a^6}\cdot\sqrt[3]{8}$.

5. Show that Exs. 1–4 are special cases of § 149 (i).

6. Find $\sqrt{5}\cdot\sqrt{3}$ correct to two decimal places (§ 118); then find $\sqrt{15}$ (*i.e.*, $\sqrt{5\cdot 3}$) correct to two decimal places, and compare results. Does this exercise illustrate any *practical* advantage in knowing that $\sqrt[n]{x}\cdot\sqrt[n]{y}=\sqrt[n]{xy}$? Explain.

7. By means of § 149 (ii), show that $\sqrt{35}\div\sqrt{7}=\sqrt{5}$, and that $\sqrt[3]{16\,a^2}\div\sqrt[3]{-2\,a}=\sqrt[3]{-8\,a}$. State § 149 (ii) in words.

8. Find (correct to two decimal places, § 118) $\sqrt{7}\div\sqrt{5}$, also $\sqrt{1.4}$ (*i.e.*, $\sqrt{7\div 5}$), and compare results. Does this exercise illustrate any *practical* advantage in knowing that $\dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}=\sqrt[n]{\dfrac{x}{y}}$?

9. Show that $\left(\dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}\right)^n=\dfrac{x}{y}$, and thus prove that $\sqrt[n]{\dfrac{x}{y}}=\dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$.

Verify that $\sqrt[np]{x}=\sqrt[n]{\sqrt[p]{x}}=\sqrt[p]{\sqrt[n]{x}}$ [cf. § 149 (iii)] when

10. $n=2,\quad p=2,\quad x=81.$ **12.** $n=4,\quad p=3,\quad x=c^{24}d^{12}.$

11. $n=5,\quad p=2,\quad x=m^{20}.$ **13.** $n=2,\quad p=3,\quad x=64.$

14. Use § 149 (iii) to find $\sqrt[6]{49}$ (*i.e.*, $\sqrt[3\cdot 2]{49}$), correct to two decimal places; also find $\sqrt[4]{144}$. How may you find the 9th root of any given number? the 8th root?

15. Prove the correctness of § 149 (iii). What do n and t represent in this principle? May nt, then, equal 11? Explain.

151. Special cases of § 149. Besides the three principles given in § 149, these four others are often useful:

(i) $$\sqrt[n]{x^n y}=x\sqrt[n]{y},$$

(ii) $$\sqrt[n]{xyz\cdots}=\sqrt[n]{x}\cdot\sqrt[n]{y}\cdot\sqrt[n]{z}\cdots,$$

(iii) $$\sqrt[n]{x^t}=(\sqrt[n]{x})^t,$$

(iv) $$\sqrt[n]{x^r}=\sqrt[nt]{x^{rt}}.$$

The correctness of these principles may be established by the method used for the proof of (i), § 149; it is easier, however, to regard them as special cases of § 149, thus:

$$\sqrt[n]{x^n y} = \sqrt[n]{x^n}\sqrt[n]{y}, \qquad [\S\ 149\ (i)$$
$$= x\sqrt[n]{y},$$

which establishes (i) above.

So, too,
$$\sqrt[n]{xyz\cdots} = \sqrt[n]{x}\cdot\sqrt[n]{yz\cdots} \qquad [\S\ 149\ (i)$$
$$= \sqrt[n]{x}\cdot\sqrt[n]{y}\cdot\sqrt[n]{z\cdots} = \text{etc.},$$

which proves (ii) above; and (iii) follows from (ii) by letting $x = y = z = \cdots$, and supposing the number of these factors to be t.

Again,
$$\sqrt[nt]{x^{rt}} = \sqrt[n]{\sqrt[t]{x^{rt}}} \qquad [\S\ 149\ (iii)$$
$$= \sqrt[n]{x^r}, \qquad [\text{since } \sqrt[t]{(x^r)^t} = x^r$$

which proves (iv).

EXERCISE CIII

1. Is $3\sqrt{5}$ equal to $\sqrt{3^2\cdot 5}$? Why? May $2\sqrt[3]{6}$ be written as $\sqrt[3]{2^3\cdot 6}$? Why? How may the coefficient of a radical be inserted under the radical sign [cf. § 151 (i)]?

2. By means of § 151 (i) show that $\sqrt{20} = 2\sqrt{5}$, and also that $\sqrt[3]{-54} = -3\sqrt[3]{2}$. How may we simplify a square root which contains a square factor? a 5th root which contains a factor raised to the 5th power?

3. By the method of § 150, show that $x\sqrt[n]{y} = \sqrt[n]{x^n y}$.

4. Verify that $\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{5} = \sqrt{30}$; find at least two decimal places in each member of the equation (cf. Ex. 6, p. 239). How find the product of several radicals of the same order?

5. By the method of § 150, show that $\sqrt[n]{x}\cdot\sqrt[n]{y}\cdot\sqrt[n]{z} = \sqrt[n]{xyz}$.

6. Find, correct to two decimal places, $(\sqrt{6})^3$, also $\sqrt{6^3}$, and compare results. How would you raise $\sqrt[3]{15}$ to the second power? to the 5th power? Explain.

7. Show, using the method of § 150, that $(\sqrt[n]{x})^t = \sqrt[n]{x^t}$.

8. Verify that $\sqrt[6]{a^{12}} = \sqrt[3]{a^6}$, and that $\sqrt[6]{a^{12}} = \sqrt{a^4}$. In each of these equations compare the exponents of a, also the root-indices.

9. Is a radical changed in value if we multiply its root-index and also the exponent of its radicand by the same factor [cf. § 151 (iv)]? if we divide both by any factor common to them? Illustrate.

10. Using § 151 (iv), show that $\sqrt[4]{9}$ (*i.e.*, $\sqrt[2\cdot 2]{3^2}$) $= \sqrt{3} = \sqrt[8]{81}$. Which is more easily computed, $\sqrt[10]{32}$, or its equal, $\sqrt{2}$?

152. Reduction of radicals to their simplest forms. A radical is said to be in its **simplest form** when the radicand is integral, when the index of the root is as small as possible, and when no factor of the radicand is a perfect power corresponding in degree with the indicated root.

The following examples illustrate the application of the foregoing principles in the reduction of radicals to their simplest form.

Ex. 1. Reduce $\sqrt{8a^3x^5}$ to its simplest form.

SOLUTION.
$$\begin{aligned}\sqrt{8a^3x^5} &= \sqrt{4a^2x^4} \cdot \sqrt{2ax} \qquad &&[\S\ 149\ (\text{i})\\ &= 2ax^2\sqrt{2ax}.\end{aligned}$$

Ex. 2. Reduce $\sqrt[6]{4a^2x^4y^4}$ to its simplest form.

SOLUTION.
$$\sqrt[6]{4a^2x^4y^4} = \sqrt[6]{(2ax^2y^2)^2} = \sqrt[3]{2ax^2y^2}. \qquad [\S\ 151\ (\text{iv})$$

Ex. 3. Reduce $\sqrt[3]{\frac{2}{5}}$ to its simplest form.

SOLUTION.
$$\begin{aligned}\sqrt[3]{\frac{2}{5}} &= \sqrt[3]{\frac{2\cdot 5^2}{5\cdot 5^2}}\\ &= \sqrt[3]{\frac{1}{5^3}\cdot 50} = \frac{1}{5}\sqrt[3]{50}. \qquad &&[\S\ 151\ (\text{i})\end{aligned}$$

EXERCISE CIV

Reduce each of the following to its simplest form:

4. $\sqrt{18}$ (*i.e.*, $\sqrt{9\cdot 2}$). **6.** $\sqrt{45}$. **8.** $\sqrt[3]{16}$.

5 $\sqrt{24}$. **7.** $\sqrt{75}$. **9.** $\sqrt[3]{-24}$.

10. $2\sqrt[3]{54}$.

11. $\sqrt[4]{32}$.

12. $\sqrt{\frac{7}{9}}$ (*i.e.*, $\sqrt{\frac{1}{9}\cdot 7}$).

13. $\sqrt{\frac{3}{16}}$.

14. $\sqrt{\frac{12}{25}}$.

15. $\sqrt[3]{-\frac{3}{8}}$.

16. $\sqrt{\frac{1}{2}}$ (cf. Ex. 3).

17. $\sqrt{\frac{5}{8}}$.

18. $10\sqrt{\frac{4}{5}}$.

19. $\sqrt[3]{\frac{1}{4}}$.

20. $2\sqrt[3]{\frac{5}{9}}$.

21. $\sqrt[4]{\frac{2}{27}}$.

22. $\sqrt[5]{-\frac{7}{8}}$.

23. $\sqrt{27\,x^2}$.

24. $\sqrt[3]{8\,m^5}$.

25. $\sqrt[6]{a^8b^{10}}$.

26. $\sqrt[4]{x^3y^8}$.

27. $\sqrt[3]{\dfrac{x^2}{y^2}}$.

28. $\sqrt{\dfrac{3\,x^2y}{z^7}}$.

29. $\sqrt[4]{\dfrac{16\,m^6}{n^6}}$.

30. $\sqrt[4]{a^2}$ (cf. Ex. 2).

31. $\sqrt[9]{a^6}$.

32. $\sqrt[6]{x^4y^2}$.

33. $\sqrt[4]{25}$.

34. $\sqrt[6]{216}$.

35. $\sqrt[10]{32\,m^5n^{10}}$.

36. $3\sqrt[4]{36\,b^2x^{11}}$.

37. $\sqrt[5]{32\,a^7e^{12}}$.

38. $\sqrt{\dfrac{2\,a^5}{9}}$.

39. $\sqrt[4]{\dfrac{ax^6}{9\,y}}$.

40. $\dfrac{x}{y}\sqrt{\dfrac{y^3}{2\,x}}$.

41. $\sqrt[4]{\dfrac{25\,a^6}{c^2}}$.

42. $\sqrt{162}$.

43. $\sqrt[8]{36}$.

44. $2\sqrt{\frac{5}{4}\,x^7}$.

45. $\sqrt{\dfrac{27\,a^3x^9}{z^{15}}}$.

46. $\frac{1}{9}\sqrt[5]{\frac{2}{3}}$.

47. $3\sqrt[6]{125\,a^3x^8}$.

48. $\sqrt{\dfrac{x+y}{x-y}}$.

49. $\sqrt[n]{a^{2n}x^{n+1}}$.

50. $\sqrt[9]{\dfrac{64\,m^6}{125}}$.

51. $6\sqrt[3]{320}$.

52. $\sqrt[5]{-486\,u^2v^8}$.

53. $\frac{4}{5}\sqrt{2\frac{5}{8}}$.

54. $\sqrt{18\,a-9}$.

55. $\sqrt[r]{a^{r+3}b^{2r}y^{r+4}}$.

56. $\sqrt{a^{2n}x^{n+5}}$.

57. $\sqrt[3]{-40\,x^{5n+8}y^{14}}$.

58. $4\sqrt[n]{a^{3n}-a^{4n}x^n}$.

59. $3\,a\sqrt{\dfrac{a-2\,bx}{2\,a}}$.

60. $\sqrt{3x^2-6xy+3y^2}$.

In the following, insert the coefficients under the radical signs:

61. $3\sqrt{7}$.

62. $5\sqrt[3]{4}$.

63. $2\sqrt[4]{6}$.

64. $-2\sqrt{8}$.

65. $\frac{3}{10}\sqrt{40}$.

66. $-4\sqrt[3]{\frac{2}{9}}$.

67. $\frac{2}{5}\sqrt{1\frac{11}{24}}$.

68. $\dfrac{x}{3}\sqrt[5]{72\,x^2}$.

69. $\dfrac{3\,a}{4}\sqrt{12\,a^2x}$.

70. $(c+1)\sqrt{5\,c}$.

71. $\dfrac{1}{ax}\sqrt[3]{a^2x(x-\frac{1}{2})}$.

72. $4\,u^3v\sqrt[n]{4\,uv^2}$.

73. $-2y^mz^p\sqrt[r]{y^2z^3}$.

153. Addition and subtraction of radicals. Similar radicals (§ 147) may evidently be added and subtracted by regarding the common radical factor as the unit of addition. The sum or difference of dissimilar radicals can, of course, only be indicated, and this is done by connecting the radicals with the proper signs (cf. § 23).

The radicals to be added or subtracted should first be reduced to their simplest forms; the following examples will illustrate the procedure.

Ex. 1. Find the sum of $\sqrt{75}$ and $3\sqrt{12}$.

Solution. Since $\sqrt{75}=\sqrt{25\cdot 3}=\sqrt{25}\cdot\sqrt{3}=5\sqrt{3}$, [§ 149 (i)

and $3\sqrt{12}=3\sqrt{4\cdot 3}=3\sqrt{4}\cdot\sqrt{3}=6\sqrt{3}$, [§ 149 (i)

therefore $\sqrt{75}+3\sqrt{12}=11\sqrt{3}$.

Ex. 2. Find the sum of $5\sqrt{18}$ and $-\sqrt{0.5}$.

Solution. Since $5\sqrt{18}=5\sqrt{9\cdot 2}=5\cdot 3\sqrt{2}=15\sqrt{2}$,

and $-\sqrt{0.5}=-\sqrt{\frac{1}{2}}=-\sqrt{\frac{1}{4}\cdot 2}=-\frac{1}{2}\sqrt{2}$,

therefore $5\sqrt{18}+(-\sqrt{0.5})=(15-\frac{1}{2})\sqrt{2}=14\frac{1}{2}\sqrt{2}$.

Ex. 3. Find the sum of $\sqrt{9x-18}$, $6\sqrt{4x+8}$, $\sqrt{36x-72}$, and $-\sqrt{25x+50}$.

Solution.
$$\begin{aligned}&\sqrt{9x-18}+6\sqrt{4x+8}+\sqrt{36x-72}-\sqrt{25x+50}\\&=3\sqrt{x-2}+12\sqrt{x+2}+6\sqrt{x-2}-5\sqrt{x+2}\\&=9\sqrt{x-2}+7\sqrt{x+2}.\end{aligned}$$

EXERCISE CV

Find the sum of:

4. $\sqrt{5}$, $7\sqrt{5}$, and $-3\sqrt{5}$.
5. $\sqrt{18}$, $\sqrt{50}$, and $\sqrt{98}$.
6. $\sqrt{12}$, $\sqrt{75}$, and $\sqrt{27}$.
7. $\sqrt{28}$, $-2\sqrt{63}$, and $\sqrt{700}$.
8. $2\sqrt{20}$, $-\frac{1}{4}\sqrt{45}$, and $\sqrt{80}$.
9. $\sqrt[3]{250}$, $\sqrt[3]{16}$, and $\sqrt[3]{54}$.
10. $\sqrt[3]{500}$, $\sqrt[3]{108}$, and $\sqrt[3]{-32}$.
11. $\sqrt[3]{cdf^4}$, $3\sqrt[3]{cdf}$, $\sqrt[3]{-8c^4df}$.

12. Find (correct to two decimal places) the value of each of the surds in Ex. 7 (cf. § 118) and add your results. How does this sum compare with that previously found for Ex. 7? Is there, then, any practical advantage in simplifying surds before adding them?

13. Find the sum of a, $2\,b$, and c; of $3\,x$, $4\,y$, $2\,x$, and $-5\,y$.

14. What is the sum of $3\sqrt{2}$ and $5\sqrt[3]{7}$? of $3\sqrt{2}$, $5\sqrt[3]{7}$, $-2\sqrt{7}$, and $\sqrt{2}$?

15. Write a rule for the addition and subtraction of radicals, providing both for those cases in which the given radicals are similar and for those in which they are dissimilar.

Simplify the following expressions as far as possible (cf. § 152), and explain your work in each case:

16. $\sqrt[3]{135}+\sqrt[3]{625}-\sqrt[3]{320}$.

17. $\sqrt[3]{40}+\sqrt{28}+\sqrt{175}+\sqrt[6]{25}$.

18. $\sqrt[3]{375}-\sqrt{44}-\sqrt[3]{192}+\sqrt{99}$.

19. $\sqrt{\frac{1}{3}}+\sqrt{75}-\sqrt{12}+\frac{2}{3}\sqrt{3}$.

20. $\sqrt{147}-\sqrt{\frac{3}{4}}+\frac{1}{3}\sqrt{3}+\frac{7}{6}\sqrt[4]{9}$.

21. $6\sqrt[3]{\frac{40}{27}}+4\sqrt[3]{\frac{10}{16}}-8\sqrt[3]{\frac{675}{320}}$.

22. $\sqrt[3]{128\,x}+\sqrt[3]{375\,x}-\sqrt[3]{54\,x}$.

23. $\sqrt{\frac{a}{x^2}}+\sqrt{\frac{a}{y^2}}-\sqrt{\frac{a}{z^2}}$.

24. $\sqrt[4]{\frac{a^2x^4}{b^2y^4}}-\sqrt{\frac{16\,ax^2}{by^2}}+\sqrt{\frac{4\,ax^2}{by^2}}$.

25. $\sqrt{(a+b)^5c}-\sqrt[6]{(a+b)^9c^3}$.

26. $\sqrt[6]{192\,x^4}-2\sqrt[6]{3\,x^4}-\sqrt[3]{5\,x}+\sqrt[3]{40\,x^4}$.

27. $\sqrt[3]{abx}+\sqrt[6]{a^2b^2x^2}-\sqrt[9]{8\,a^3b^3x^3}$.

28. $\sqrt{3\,x^3+30\,x^2+75\,x}-\sqrt{3\,x^3-6\,x^2+3\,x}$.

29. $\sqrt{5\,a^5+30\,a^4+45\,a^3}-\sqrt{5\,a^5-40\,a^4+80\,a^3}$.

30. $\sqrt{50}+\sqrt[6]{9}-4\sqrt{\frac{1}{2}}+\sqrt[3]{24}+\sqrt[9]{27}-\sqrt[4]{64}$.

31. $\sqrt{\frac{2}{3}}+6\sqrt{\frac{5}{4}}-\frac{1}{5}\sqrt{18}+\sqrt[4]{36}-\sqrt[8]{\frac{16}{81}}+\sqrt[6]{125}-\sqrt{\frac{2}{25}}$.

32. $\sqrt{a^3-a^2x}-\sqrt{ax^2-x^3}-\sqrt{(a+x)(a^2-x^2)}$.

154. Reduction of radicals to the same order. By (iv) of § 151,

$$\sqrt[3]{5}=\sqrt[3\cdot4]{5^4}=\sqrt[12]{5^4}=\sqrt[12]{625},$$

and

$$\sqrt[4]{7}=\sqrt[4\cdot3]{7^3}=\sqrt[12]{7^3}=\sqrt[12]{343},$$

i.e., the radicals $\sqrt[2]{5}$ and $\sqrt[4]{7}$ are equivalent, respectively, to $\sqrt[12]{625}$ and $\sqrt[12]{343}$; and these last two radicals are of the same order, viz., the twelfth.

Moreover, it is evident that, by a similar application of § 151 (iv), any two or more radicals whatever may be reduced to equivalent radicals of the same order. This new order must, of course, be some common multiple (preferably the L. C. M.) of the orders of the given radicals.

EXERCISE CVI

1. Is $\sqrt[6]{x^2}$ equal to $\sqrt[3]{x}$? Is $\sqrt[5]{3a^2x^4}$ equal to $\sqrt[10]{9a^4x^8}$? Using the method of § 150, prove the correctness of your answer to each of these questions. Compare also § 151 (iv).

2. Reduce $\sqrt[8]{25m^4z^6}$ and $\sqrt[12]{8a^9b^6x^3}$ to equivalent radicals of the 4th order, and explain (cf. Exs. 8–10, p. 241).

Reduce to equivalent radicals of the order indicated, and explain your work:

3. $\sqrt[3]{x^2}$, 9th order.

4. $\sqrt[3]{3a^4b}$, 6th order.

5. $\sqrt[5]{2st^2}$, 10th order.

6. $\sqrt{2st^2}$, 10th order.

7. $3ax$, 4th order.

8. $\sqrt[3]{2m^2n}$, 12th order.

9. $\sqrt[4]{a^2n^3x^2}$, 12th order.

10. $\sqrt{a^2n^3x^2}$, 8th order.

11. Express as equivalent radicals of the 6th order: $\sqrt[3]{m}$, $\sqrt[12]{9m^6n^2}$, and mn.

Reduce the following to equivalent radicals of the same order:

12. $\sqrt{5}$ and $\sqrt[3]{11}$.

13. $\sqrt[4]{7}$ and $\sqrt{3}$.

14. $\sqrt{2a}$ and $\sqrt[5]{3a^2}$.

15. $\sqrt{6}$ and 3.

16. $\sqrt{p}$, $\sqrt[4]{5p^3}$, and $\sqrt[8]{7p^2}$.

17. $\sqrt[6]{10}$, $\sqrt{2}$, and $\sqrt[3]{5}$.

18. $\sqrt[3]{2x^2y}$, $\sqrt[4]{3x^3}$ and xy.

19. $\sqrt{14}$, $\sqrt[3]{25}$, and $\sqrt[6]{95}$.

20. $3\sqrt{3}$, $5\sqrt{2}$, and $2\sqrt[4]{20}$.

21. $2\sqrt{3}$, $3\sqrt[3]{2}$, and $2\sqrt[6]{39}$.

22. $3\sqrt{ab}$, $a\sqrt[3]{2x^2}$, and $\sqrt[6]{a^2b^3}$.

23. $x^3\sqrt[4]{2y}$, $\sqrt{xy}$, and $2\sqrt[3]{m^2z}$.

24. $\sqrt[3]{x}$, $5x\sqrt{y}$, and $\sqrt[n]{x^2y^2}$.

25. $\sqrt[4]{a^2+b^2}$, and $a\sqrt{a-b}$.

26. Can the radicals in Ex. 22 be reduced to equivalent radicals of the 6th order? of the 12th order? of the 9th order? Give the reasons for your answer in each case.

27. What is the lowest *common* order to which you can reduce the radicals in Ex. 23? those in Ex. 24? in Ex. 25?

28. By first reducing $\sqrt[3]{15}$ and $\sqrt{6}$ to the same order, show that $\sqrt[3]{15}$ is greater than $\sqrt{6}$.

29. Which is greater, $\sqrt{5}$ or $\sqrt[3]{12}$? Explain your answer (cf. Ex. 28).

30. Which is greater, $3\sqrt{10}$ or $2\sqrt[3]{100}$?

Hint. First insert coefficients under radical signs (cf. Ex. 1, p. 240).

Arrange the following radicals in order of magnitude:

31. $3\sqrt{3}$, $5\sqrt{2}$, and $2\sqrt[4]{20}$. **32.** $2\sqrt[6]{39}$, $3\sqrt[3]{2}$, and $2\sqrt{3}$.

155. Product of monomial radicals. If two or more radicals are of the same order, their product may be written down immediately by § 151 (ii); while if they are of different orders, they should first be reduced to equivalent radicals of the same order (§ 154).

Ex. 1. Multiply $\sqrt[3]{5}$ by $\sqrt{2}$.

Solution. The L. C. M. of the orders of the given radicals is 6, and by § 154 $\sqrt[3]{5}=\sqrt{5^2}$,

and $\sqrt{2}=\sqrt[6]{2^3}$;

therefore $\sqrt[3]{5}\cdot\sqrt{2}=\sqrt[6]{5^2}\cdot\sqrt[6]{2^3}=\sqrt[6]{5^2\cdot 2^3}$ [§ 149 (i)

$=\sqrt[6]{200}$.

Ex. 2. Find the product of $4b\sqrt{ax}$ and $\sqrt[3]{5a^2}$, and simplify the result.

Solution. As in Ex. 1, $4b\sqrt{ax}=4b\sqrt[6]{a^3x^3}$,

and $\sqrt[3]{5a^2}=\sqrt[6]{5^2a^4}$;

therefore $4b\sqrt{ax}\cdot\sqrt[3]{5a^2}=4b\sqrt[6]{a^3x^3}\cdot\sqrt[6]{5^2a^4}$

$=4b\sqrt[6]{5^2a^7x^3}$ [§ 149 (i)

$=4ab\sqrt[6]{25ax^3}$. [§ 151 (i)

EXERCISE CVII

Find the following products, and simplify the results:

3. $\sqrt{3} \cdot \sqrt{6}$.

4. $2\sqrt{15} \cdot \sqrt{5}$.

5. $\sqrt[4]{27} \cdot \sqrt[4]{6}$.

6. $\sqrt[3]{\frac{1}{2}} \cdot \sqrt[3]{-\frac{1}{4}}$.

7. $2\sqrt[4]{\frac{1}{8}} \cdot \sqrt[4]{1\frac{3}{8}}$.

8. $\sqrt[7]{x} \cdot \sqrt[7]{-3\,x^{15}}$.

9. $\sqrt{\dfrac{3\,x}{2\,y}} \cdot \sqrt{\dfrac{x}{y}}$.

10. $\frac{11}{14}\sqrt[3]{7\,m^2} \cdot \sqrt[3]{49\,m^5n^2}$.

11. $(8-2\sqrt{15}) \cdot 2\sqrt{6}$.

12. $\left(\sqrt{\dfrac{a}{b}}+\sqrt{\dfrac{c}{a}}\right) \cdot \sqrt{\dfrac{2\,a}{b}}$.

13. How may we find the product of two or more radicals which are of the same order?

14. How may we find the product of two or more radicals of different orders? Illustrate, using the surds $\sqrt{3}$ and $\sqrt[3]{2}$.

Find the following products, and simplify the results:

15. $\sqrt{ab} \cdot \sqrt[3]{bc}$.

16. $\sqrt{3} \cdot 3\sqrt[4]{3}$.

17. $\sqrt{2} \cdot \sqrt[3]{4}$.

18. $2\sqrt[4]{5} \cdot 7\sqrt[6]{10}$.

19. $\sqrt[5]{\frac{1}{2}} \cdot \sqrt[10]{(\frac{1}{2})^6}$.

20. $\sqrt[3]{\dfrac{p^2q^3}{r^4}} \cdot \sqrt[4]{\dfrac{p^3q^2}{r}}$.

21. $\sqrt[3]{4\,a^2} \cdot \sqrt{8\,a^3}$.

22. $\sqrt[n]{a^t} \cdot \sqrt[m]{a^s}$.

23. $\sqrt[5]{\frac{3}{8}} \cdot \sqrt[5]{2\frac{3}{8}}$.

24. $2\sqrt{2} \cdot \sqrt[10]{256}$.

25. $\sqrt[3]{2} \cdot \sqrt[6]{\frac{1}{3}} \cdot \sqrt[8]{3}$.

26. $\sqrt{x^3y^2} \cdot \sqrt{12\,x} \cdot \sqrt{75\,xy^2}$.

27. $\sqrt{2\,ab} \cdot \sqrt[3]{abc} \cdot \sqrt[4]{4\,a^2b^2}$.

28. $\sqrt[3]{9(c-k)^2} \cdot \frac{1}{9}\sqrt{3(c-k)}$.

29. $(\sqrt{72}+\sqrt[3]{12}-3\sqrt{98}) \cdot \sqrt{2}$.

30. $\left(\sqrt{\dfrac{m}{n}}-\sqrt{\dfrac{n}{p}}+\sqrt[3]{\dfrac{p}{m}}\right) \cdot \sqrt{\dfrac{m}{n}}$.

156. Multiplication of polynomials containing radicals. The product of two polynomials containing radicals is obtained by multiplying each term of the multiplicand by each term of the multiplier and adding the partial products, just as in the case of rational polynomials.

Ex. 1. Multiply $5\sqrt{2}-2\sqrt{3}$ by $3\sqrt{2}+4\sqrt{3}$.

SOLUTION.

$$\begin{array}{r} 5\sqrt{2}-2\sqrt{3} \\ 3\sqrt{2}+4\sqrt{3} \\ \hline 30-6\sqrt{6} \\ +20\sqrt{6}-24 \\ \hline 30+14\sqrt{6}-24=6+14\sqrt{6}. \end{array}$$

Ex. 2. Expand $(2\sqrt{3}-\sqrt[3]{2})^2$ by the binomial theorem.

SOLUTION. $(2\sqrt{3}-\sqrt[3]{2})^2=(2\sqrt{3})^2-2(2\sqrt{3})\sqrt[3]{2}+(\sqrt[3]{2})^2$
$=12-4\sqrt[6]{108}+\sqrt[3]{4}.$

EXERCISE CVIII

Multiply, and express your results in the simplest form:

3. $\sqrt{5}-5$ by $\sqrt{5}+1$.
4. $2\sqrt{7}-3$ by $2\sqrt{7}+4$.
5. $5\sqrt{11}+4$ by $5\sqrt{11}-4$.
6. $\sqrt{c}+\sqrt{2d}$ by $\sqrt{c}-\sqrt{2d}$.
7. $(\sqrt{ac}-\sqrt{3ab})^2$.
8. $(3\sqrt{\frac{1}{6}}+5\sqrt{\frac{1}{2}})^2$.
9. $(a-ab\sqrt{2}+b^2)^2$.
10. $\sqrt[3]{3}-3\sqrt[3]{9}$ by $\sqrt[3]{3}+3\sqrt[3]{9}$.
11. $(\sqrt[3]{9}-4\sqrt[3]{6})^2$.
12. $\left(\frac{a}{b}-\sqrt[4]{\frac{b}{c}}\right)$ by $\left(\frac{a}{b}-\sqrt[4]{\frac{b}{c}}\right)$.
13. $\sqrt{2}+\sqrt{3}+\sqrt{6}$ by $\sqrt{2}+4\sqrt{3}$.
14. $x-\sqrt{xyz}+yz$ by $\sqrt{x}+\sqrt{yz}$.
15. $\sqrt{2m}-\sqrt[3]{3x^2}$ by $\frac{m}{2}-\sqrt[3]{\frac{x}{4}}$.
16. $3\sqrt[3]{3}-\sqrt{2}$ by $5\sqrt[3]{4}+\sqrt{2}$.
17. $2\sqrt{3}-\sqrt[3]{2}$ by $2\sqrt{3}-\sqrt[3]{4}$.
18. $\sqrt{8-2\sqrt{7}}$ by $\sqrt{8+2\sqrt{7}}$.
19. $\sqrt{10}-3\sqrt{5}-4\sqrt{3}$ by $4\sqrt{3}-3\sqrt{5}$.
20. $\sqrt{xy}+2\sqrt{xz}-\sqrt{yz}$ by $\sqrt{xy}-3\sqrt{xz}$.
21. $\sqrt[3]{m^2n^2}+\sqrt[3]{mn}$ by $\sqrt[3]{mn^2}+\sqrt[3]{m^2n}$.
22. $2\sqrt[5]{x^4y^3}-3\sqrt[5]{4x^4z^3}$ by $\sqrt[5]{16x^3y^2}+\sqrt[5]{x^4}$.
23. $\sqrt{x^m}-\sqrt{10x^{n+1}}+\sqrt{x^2}$ by $\sqrt{5x^m}+\sqrt{2x^{n+3}}$.
24. $\sqrt[m]{(x+y)^4}-\sqrt[m]{(x+y)^{m+2}}$ by $\sqrt[m]{(x+y)^{2m+4}}+\sqrt[m]{(x+y)^2}$.
25. $x+\frac{p}{2}-\sqrt{\frac{p^2-4q}{4}}$ by $x+\frac{p}{2}+\sqrt{\frac{p^2+4q}{4}}$.

157. Rationalizing factors.* **Conjugate surds.** The factor by which a given surd (radical) must be multiplied in order to obtain a rational product is called its **rationalizing factor.** Thus, of the surds $\sqrt[3]{5a^2}$ and $\sqrt[3]{25a}$, each is the rationalizing factor of the other, since their product, $5a$, is rational; the same is true of the surds $2\sqrt{a}-\sqrt{3}$ and $2\sqrt{a}+\sqrt{3}$. (Why?)

Of two such binomial quadratic surds as $2\sqrt{a}-\sqrt{3}$ and $2\sqrt{a}+\sqrt{3}$, which differ only in the sign of one term, each is called the **conjugate** of the other. Moreover, since the product of any two conjugate quadratic surds is rational (§ 53), therefore each of them is the rationalizing factor of the other.

EXERCISE CIX

1. Is $\sqrt{2a}$ the rationalizing factor of $\sqrt{2a}$? Why? Show that $\sqrt[4]{8k^2l}$ and $2\sqrt{3}-\sqrt{5}$ are the rationalizing factors of $\sqrt[4]{2k^2l^3}$ and $2\sqrt{3}+\sqrt{5}$, respectively.

2. Is $\sqrt{5}-2\sqrt{3}$ the rationalizing factor of $2\sqrt{3}+\sqrt{5}$? Explain. Are these surds conjugate to each other?

Find the rationalizing factor of:

3. $\sqrt{3}$.

4. $\sqrt{7}$.

5. $2\sqrt{10}$.

6. $\sqrt[4]{27}$.

7. $\sqrt[3]{4ax^2}$.

8. $\sqrt[3]{-\frac{1}{2}}$.

9. $5\sqrt[7]{x^2z^3}$.

10. $\sqrt[4]{\frac{3}{4}}$.

11. $\dfrac{2}{x}\sqrt[6]{\dfrac{x}{3y^2}}$.

12. $\sqrt[3]{\dfrac{a+b}{18}}$.

13. $\sqrt{2}-\sqrt{7}$.

14. $4+5\sqrt{3}$.

15. $2\sqrt{3}+\sqrt{8}$.

16. $3a-\sqrt{5x}$.

17. $\sqrt{5x}-\sqrt{2ay}$.

18. $\sqrt{a^3}+2\sqrt{3b}$.

19. $\sqrt{\dfrac{x}{3}}-\sqrt{\dfrac{yz^2}{10}}$.

20. $\dfrac{a}{b}\sqrt{a}-c\sqrt{\dfrac{c}{d}}$.

21. $\sqrt{\dfrac{14x}{a}}+\dfrac{3}{4}\sqrt{ax^3}$.

22. How may we find the rationalizing factor of *any* binomial quadratic surd? Why? Does the same method answer for a binomial *cubic* surd? Explain.

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* See also § 177.

158. Division of monomial radicals. If the dividend and divisor are of the same order, their quotient may be written down immediately by § 149 (ii), while if they are of different orders, they should first be reduced to equivalent radicals of the same order (§ 154).

E.g., to divide $\sqrt[6]{4ax^3y^2}$ by $\sqrt[4]{2a^3x}$, we proceed thus:

$$\frac{\sqrt[6]{4ax^3y^2}}{\sqrt[4]{2a^3x}} = \frac{\sqrt[12]{16a^2x^6y^4}}{\sqrt[12]{8a^9x^3}} \qquad [§ 154$$

$$= \sqrt[12]{\frac{16a^2x^6y^4}{8a^9x^3}} \qquad [§ 149 \text{ (ii)}$$

$$= \sqrt[12]{\frac{2x^3y^4}{a^7}} = \frac{1}{a}\sqrt[12]{2a^5x^3y^4}. \qquad [§ 152$$

159. Division of polynomials containing radicals. If the divisor is a monomial, then, manifestly, the quotient may be obtained by dividing each term of the dividend by the divisor.

Ex. 1. Divide $3\sqrt{2} + 4\sqrt{3}$ by $\sqrt{2}$.

SOLUTION. $$\frac{3\sqrt{2} + 4\sqrt{3}}{\sqrt{2}} = 3 + 4\sqrt{\frac{3}{2}} = 3 + 2\sqrt{6}. \qquad [§ 152$$

We may also solve Ex. 1 as follows:

$$\frac{3\sqrt{2} + 4\sqrt{3}}{\sqrt{2}} = \frac{(3\sqrt{2} + 4\sqrt{3})\sqrt{2}}{(\sqrt{2})^2} = \frac{6 + 4\sqrt{6}}{2} = 3 + 2\sqrt{6},$$

i.e., we may, before dividing, multiply both dividend and divisor by the rationalizing factor (§ 157) of the latter. This method is known as "division by means of rationalizing the divisor"; in many examples it is easier than the one used in the first solution above (cf. Exs. 28, 29, and 37 below).

EXERCISE CX

Find the following indicated quotients and simplify each:

2. $\sqrt{20} \div \sqrt{5}$.
3. $\sqrt{216} \div \sqrt{12}$.
4. $6\sqrt{5} \div \sqrt{40}$.
5. $\sqrt{pq} \div \sqrt{3qr}$.

6. $\sqrt[3]{c^2d} \div \sqrt[3]{cy}$.

7. $2\sqrt[3]{54} \div \sqrt[3]{216}$.

8. $3\sqrt[3]{18} \div 15\sqrt[3]{\frac{3}{4}}$.

9. $\sqrt[5]{(a-b)^2} \div \sqrt{(a+b)^2}$.

10. $2\sqrt{6} \div \sqrt[4]{3}$.

11. $2\sqrt[3]{12} \div \sqrt{8}$.

12. $10 \div \sqrt{5}$.

13. $6 \div 3\sqrt[3]{4}$.

14. $\sqrt{\frac{4}{3}} \div 7\sqrt[3]{\frac{2}{3}}$.

15. $\sqrt[3]{5a^2x^3} \div \sqrt{2ax}$.

16. $\dfrac{4x}{y} \div \sqrt[3]{\dfrac{x}{y^2}}$.

17. $a\sqrt[9]{4x^2y^2} \div 2b\sqrt[4]{2xy}$.

18. $3a\sqrt[2n]{2x^{2n+1}} \div 2b\sqrt[n]{3x^{n+2}}$.

19. Find the value of $1 \div \sqrt{2}$, correct to two decimal places: (1) by finding $\sqrt{2}$ and dividing, and (2) by first rationalizing the divisor. Which is the easier process?

20. Find (correct to two decimal places) the value of:

$$\frac{1}{\sqrt{3}};\ \frac{3}{\sqrt{2}};\ \frac{\sqrt{3}}{\sqrt{2}};\ \text{and}\ \frac{12}{\sqrt{6}}.\quad \text{(Cf. Ex. 19.)}$$

Find the following indicated quotients and simplify each:

21. $(\sqrt{15} - \sqrt{3}) \div \sqrt{3}$.

22. $(\sqrt{6} + 2\sqrt{3}) \div \sqrt{2}$.

23. $(4 - 7\sqrt{5}) \div \sqrt{6}$.

24. $(x\sqrt{y^2z} - x^3yz^2) \div \sqrt{xyz}$.

25. $(\sqrt[3]{a^2b} - \sqrt[3]{b^2c} + \sqrt[3]{c^2a}) \div \sqrt[3]{abc}$.

26. $(5\sqrt[3]{12} - 2\sqrt{6} + 4) \div \sqrt[3]{4}$.

27. In which of the above exercises is division by means of rationalizing the divisor easier than direct division?

Divide by means of rationalizing the divisor, and simplify:

28. $\dfrac{1}{\sqrt{3} - \sqrt{2}}$.

29. $\dfrac{5}{3\sqrt{6} - 7}$.

30. $\dfrac{2 + \sqrt{3}}{2 - \sqrt{3}}$.

31. $\dfrac{c^2}{\sqrt{c} - \sqrt{d}}$.

32. $\dfrac{8 + 2\sqrt{15}}{\sqrt{3} + \sqrt{5}}$.

33. $\dfrac{27 - 4\sqrt{7}}{2\sqrt{7} - 1}$.

34. $\dfrac{3\sqrt{2} - 4\sqrt{5}}{2\sqrt{3} + \sqrt{7}}$.

35. $\dfrac{a + \sqrt{a^2 + x}}{a - \sqrt{a^2 + x}}$.

36. $\dfrac{\sqrt{s+t} - \sqrt{s-t}}{\sqrt{s+t} + \sqrt{s-t}}$.

37. If the result in Ex. 26 were wanted correct to 3 decimal places, show in detail that it is far simpler first to rationalize the divisor than to extract roots and divide by the ordinary arithmetical method. Is this true in Ex. 28? in Ex. 33?

38. What is the product of $(2+\sqrt{3})-\sqrt{5}$ by $(2+\sqrt{3})+\sqrt{5}$? of this result by $2-4\sqrt{3}$? What, then, is the rationalizing factor of $2+\sqrt{3}-\sqrt{5}$? Divide 2 by $2+\sqrt{3}+\sqrt{5}$; also by $1+\sqrt{3}-\sqrt{2}$.

160. Powers of surds. Roots of monomial surds. Powers of surds, being merely products of two or more *equal* surd factors, may be found by the method of § 155; and roots of monomial surds may be found by § 149 (iii).

Ex. 1. Find the fifth power of $\sqrt[3]{2a^2}$.

SOLUTION. $(\sqrt[3]{2a^2})^5 = \sqrt[3]{(2a^2)^5}$ [§ 151 (iii)

$= \sqrt[3]{2^5 a^{10}} = 2a^3\sqrt[3]{4a}$. [§ 152

Ex. 2. Extract the square root of $\sqrt[3]{4a^2x}$.

SOLUTION. $\sqrt{\sqrt[3]{4a^2x}} = \sqrt[6]{4a^2x}$. [§ 149 (iii)

This result may also be written in either of the following forms:

$\sqrt[3]{\sqrt{4a^2x}}$ or $\sqrt[3]{2a\sqrt{x}}$. [§ 149 (iii)

EXERCISE CXI

Simplify:

3. $(\sqrt{7c^3})^2$.
4. $(2\sqrt[3]{ab})^2$.
5. $(\sqrt[5]{6mn^4})^2$.
6. $(\frac{1}{2}\sqrt{\frac{2}{7}})^3$.
7. $\left(\sqrt[3]{\frac{x}{y}}\right)^4$.
8. $(\frac{3}{8}\sqrt[5]{16})^2$.
9. $\left(\sqrt[3]{\frac{r^2s}{6}}\right)^6$.
10. $(-\sqrt{2}\sqrt[12]{m})^3$.
11. $(\sqrt[m]{u^2v^3w^m})^{2m}$.
12. $(\sqrt[m]{u^{2m}v^{m+1}})^{2n}$.
13. $(9+\sqrt{5})^2$.
14. $(\sqrt{7}-\sqrt{2})^2$.
15. $(\sqrt{2xy}+\sqrt{3yz})^2$.
16. $(5\sqrt{3}-3\sqrt{2})^2$.
17. $(\sqrt{m}-\sqrt{n})^3$.
18. $(1-2\sqrt{3})^3$.
19. $(\sqrt[3]{m^2}-\sqrt[3]{mn})^2$.
20. $(\sqrt{2}-3\sqrt[3]{3})^2$.

Express each of the following by means of a single radical sign, and simplify:

21. $\sqrt{\sqrt{a}}$.
22. $\sqrt{\sqrt[3]{3cd^2}}$.
23. $\sqrt{\sqrt[5]{25m^2n^2}}$.
24. $\sqrt[3]{-27\sqrt{x}}$.
25. $\sqrt[3]{-\frac{1}{8}\sqrt{m^5}}$.
26. $\sqrt{\sqrt{(r+s)^{10n}}}$.
27. $\sqrt{\frac{1}{16}\sqrt[7]{x^{2n}y}}$.
28. $\sqrt[n]{\sqrt{5x^{2m+2}}}$.
29. $\sqrt[4]{\sqrt[3]{\frac{e^2}{8(e-f)}}}$.

161. An important property of quadratic surds. Neither the sum nor the difference of two dissimilar *quadratic* surds (§ 147) can be a rational number; for, if possible, let

$$\sqrt{x} \pm \sqrt{y} = r, \tag{1}$$

$\sqrt{x}$ and $\sqrt{y}$ being dissimilar surds, and r rational, and not zero.

From Eq. (1) $\qquad \pm\sqrt{y} = r - \sqrt{x}, \qquad (2)$

whence, squaring, $\qquad y = r^2 - 2r\sqrt{x} + x, \qquad (3)$

and, solving for $\sqrt{x}$, $\qquad \sqrt{x} = \dfrac{r^2 + x - y}{2r}$;

i.e., if Eq. (1) were true, then the surd $\sqrt{x}$ would equal the rational number $\dfrac{r^2+x-y}{2r}$, which is impossible; hence Eq. (1) cannot be true.

From what has just been shown it at once follows that *if $x + \sqrt{y} = a + \sqrt{b}$, where x and a are rational, and $\sqrt{y}$ and $\sqrt{b}$ are quadratic surds, then $x = a$ and $y = b$.*

For, if $\qquad x + \sqrt{y} = a + \sqrt{b}$,

then $\qquad \sqrt{y} - \sqrt{b} = a - x$;

which, by the above proof, can be true only if each member is zero, *i.e.*, if $a = x$ and $\sqrt{y} = \sqrt{b}$. In other words, the equation $x + \sqrt{y} = a + \sqrt{b}$ is equivalent to the two equations $x = a$ and $y = b$.

162. Square roots of binomial surds. Some binomial quadratic surds are exact squares; the following examples show how to extract the square root of such surds.

Ex. 1. Extract the square root of $8 + \sqrt{60}$.

Solution. If $8 + \sqrt{60}$ is the square of a binomial surd, let $\sqrt{x} + \sqrt{y}$ represent that surd, *i.e.*, let

$$\sqrt{8 + \sqrt{60}} = \sqrt{x} + \sqrt{y},$$

then, squaring, $\qquad 8 + \sqrt{60} = x + 2\sqrt{xy} + y = x + y + 2\sqrt{xy}$;

therefore $\qquad 8 = x + y$ and $\sqrt{60} = 2\sqrt{xy}$, $\qquad$ [§ 161

and combining these last two equations (after squaring the second) easily leads (§ 131) to the solution

$$x = 3 \text{ and } y = 5;$$

therefore $$\sqrt{8 + \sqrt{60}} = \sqrt{3} + \sqrt{5},$$

as is easily verified by squaring the expression $\sqrt{3} + \sqrt{5}$.

NOTE. This example might also have been solved by inspection; for, writing $8 + \sqrt{60}$ in the form $8 + 2\sqrt{15}$, and then comparing it with $(\sqrt{x} + \sqrt{y})^2$, *i.e.*, with $x + y + 2\sqrt{xy}$, we see that we have only to find two numbers whose sum is 8 and whose product is 15, and take the sum of their square roots as the required root.

Ex. 2. By inspection, find the square root of $18 - 6\sqrt{5}$.

SOLUTION. Writing $18 - 6\sqrt{5}$ in the form $18 - 2\sqrt{45}$, we see that we need to find two numbers whose sum is 18 and whose product is 45, and take the *difference* of their square roots. These numbers are evidently 3 and 15, hence

$$\sqrt{18 - 2\sqrt{45}} = \sqrt{3} - \sqrt{15}.$$

EXERCISE CXII

Find (by inspection where practicable) the square root of each of the following expressions, and check your results:

3. $4 + 2\sqrt{3}$.

4. $16 + 2\sqrt{15}$.

5. $12 + 8\sqrt{5}$.

6. $17 - 12\sqrt{2}$.

7. $27 - 4\sqrt{35}$.

8. $30 - 20\sqrt{2}$.

9. $39 - 12\sqrt{3}$.

10. $47 - 12\sqrt{11}$.

11. $63 + 24\sqrt{5}$.

12. $8xy - 4xy\sqrt{3}$.

13. $e + 4f - 4\sqrt{ef}$.

14. $2m + 9n - 6\sqrt{2mn}$.

15. $146 - 56\sqrt{6}$.

16. $m + 2\sqrt{m}$.

17. $a - \sqrt{e}$.

18. In the first solution of Ex. 1 above, why does $x + y = 8$, and $2\sqrt{xy} = \sqrt{60}$?

19. If the numerical value of $\sqrt{21 + 8\sqrt{5}}$ is required, is it easier to find first the binomial whose square is $21 + 8\sqrt{5}$, or to begin by extracting the square root of 5? Explain. Also answer the question if $12 + 3\sqrt{5}$ is substituted for $21 + 8\sqrt{5}$.

163. Irrational equations. Equations which contain indicated roots of the *unknown* numbers are called **irrational equations** (also **radical equations**). Thus $6\sqrt{x} - 25x + 88 = 0$, $\sqrt{x+1} + x = 8$, $\dfrac{x-6}{\sqrt{x}} + 1 = 0$, and $3 + \frac{1}{2}\sqrt{x} = \sqrt[3]{x^2 - 1}$ are irrational equations, but such an equation as $x - \sqrt{3} = 5x$ is rational.

The solution of irrational equations is illustrated by the following examples:

Ex. 1. Solve the equation $\sqrt{x+1} + x = 11$.

Solution. On transposing, the given equation becomes

$$\sqrt{x+1} = 11 - x,$$

whence, squaring both members (Ax. 3),

$$x + 1 = 121 - 22x + x^2,$$

i.e., $$x^2 - 23x + 120 = 0,$$

whence (§ 126), $$x = 15 \text{ or } 8;$$

and, on substitution, it is found that 15 satisfies the given equation if $\sqrt{x+1}$ means the *negative* value of this root, while 8 satisfies it if the *positive* value of this root is intended.

Ex. 2. Solve the equation $\sqrt{5x+1} - \sqrt{x+2} = 3$.

Solution. On transposing, the given equation becomes

$$\sqrt{5x+1} = 3 + \sqrt{x+2},$$

whence, squaring both members (Ax. 3),

$$5x + 1 = 9 + 6\sqrt{x+2} + x + 2,$$

i.e., $$4x - 10 = 6\sqrt{x+2};$$

whence, dividing through by 2, then squaring and simplifying,

$$4x^2 - 29x + 7 = 0,$$

from which (§ 126), $$x = 7 \text{ or } \tfrac{1}{4}.$$

On substitution it is found that the given equation is satisfied by $x = 7$ if each radical is regarded as positive, and by $x = \frac{1}{4}$ if $\sqrt{5x+1}$ is taken as positive and $\sqrt{x+2}$ as negative.

EXERCISE CXIII

Solve the following equations, and show what restrictions, if any, must be made on the signs of the radicals in order that your results shall be roots:

3. $\sqrt{2x+6}=4.$

4. $\sqrt{4x+5}=7.$

5. $\sqrt{3x-8}=\sqrt{x}.$

6. $\sqrt[3]{x^2-4x}=2\sqrt[3]{4}.$

7. $\sqrt[3]{x+5c^3}=-2c.$

8. $\sqrt{5-x}=x-5.$

9. $x+\sqrt{x}=4x-4\sqrt{x}.$

10. $y+\sqrt{y}-20=0.$

11. $\sqrt{4y+17}+\sqrt{y+1}=4.$

12. $\sqrt{25-6b}=8-\sqrt{25+6b}.$

13. $c\sqrt{x}-d\sqrt{x}=c^2+d^2-2cd.$

14. $\sqrt{x+1}+\dfrac{1}{\sqrt{x+1}}=2.$

15. $\sqrt{s-7}-\dfrac{5}{\sqrt{s+7}}=0.$

16. $\sqrt{3+x}+\sqrt{x}=\dfrac{5}{\sqrt{x}}.$

17. $\sqrt{m+5}+\sqrt{m-8}=\sqrt{3}.$

18. $\sqrt{1+s\sqrt{s^2+12}}=1+s.$

19. $\dfrac{\sqrt{v-3}}{\sqrt{v+1}}=\dfrac{\sqrt{v-4}}{\sqrt{v-2}}.$

20. $\dfrac{\sqrt{t+1}-\sqrt{t-1}}{\sqrt{t+1}+\sqrt{t-1}}=\dfrac{1}{2}.$

21. $\dfrac{\sqrt{3x+1}+\sqrt{3x}}{\sqrt{3x+1}-\sqrt{3x}}=2.$

22. $\dfrac{\sqrt{x}-2}{\sqrt{x}+3}=\dfrac{\sqrt{x}+1}{\sqrt{x}+2}.$

23. $r+\sqrt{r^2-c^2}=\dfrac{c^2}{\sqrt{r^2-c^2}}.$

24. $\dfrac{a-\sqrt{x^2-a^2}}{a+\sqrt{x^2-a^2}}=-\dfrac{\sqrt{3}-1}{\sqrt{3}+1}.$

25. $\sqrt{4x+1}-\sqrt{x+3}=\sqrt{x-2}.$

26. $\sqrt{x+a}+\sqrt{x+b}=\sqrt{2x+a+b}.$

27. $\sqrt{x+3}+\sqrt{4x+1}=\sqrt{10x+4}.$

28. $\dfrac{3x-\sqrt{x^2-8}}{x-\sqrt{x^2-8}}=x+\sqrt{x^2-8}.$

29. $y^2-y-\sqrt{y^2-y+4}=8$ (cf. Ex. 18, p. 206).

30. $a+10=2\sqrt{a+10}+5.$

31. How many roots has the equation in Ex. 27, if the radicals are unrestricted in sign? How many, if each radical must be taken positively?

CHAPTER XV

IMAGINARY NUMBERS *

164. Definitions. In the solution of quadratic equations, and elsewhere, such numbers as $\sqrt{-5}$ and $6-\sqrt{-10}$ frequently present themselves; these numbers cannot be expressed, even approximately, as the quotient of two integers (cf. § 146).

Numbers of the form $\sqrt{-b}$, where b represents a positive number, are called **pure imaginary numbers**, while numbers of the form $a \pm \sqrt{-b}$ are called **complex imaginary numbers, or complex numbers.** Two complex numbers are said to be **conjugate** if they differ only in the signs of their imaginary terms. Thus, $\sqrt{-5}$, $2\sqrt{-6}$, and $\sqrt{-\frac{1}{3}}$ are pure imaginary numbers, while $2-\sqrt{-3}$, $7-2\sqrt{-5}$, and $7+2\sqrt{-5}$ are complex numbers, the last two being conjugates of each other.

From the definition of the symbol $\sqrt[n]{a}$ (§ 113) it follows that

$$(\sqrt{-b})^2 = -b; \tag{1}$$

and by the method of proof used in § 150, it is easily shown that

$$\sqrt{-b} = \sqrt{b} \cdot \sqrt{-1}. \tag{2}$$

REMARK. The second member of Eq. (2) may be regarded as a **standard form**; and it will be found that operations with imaginary numbers are usually much simplified by first reducing such numbers to this standard form. The symbol $\sqrt{-1}$ is often called the **imaginary unit**, and is represented by i.

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* Teachers who prefer less work in imaginaries than is here given may omit §§ 166–168; the entire chapter should be read if time permits.

165. Powers of $\sqrt{-1}$, i.e., of i. As a particular case of Eq. (1), § 164, we have

$$(\sqrt{-1})^2 = -1, \qquad i.e., i^2 = -1;$$

similarly $(\sqrt{-1})^3 = (\sqrt{-1})^2 \cdot \sqrt{-1} = -\sqrt{-1}$, " $i^3 = -i$,

$$(\sqrt{-1})^4 = (\sqrt{-1})^3 \cdot \sqrt{-1} = 1, \qquad \text{"} \quad i^4 = 1,$$

$$(\sqrt{-1})^5 = (\sqrt{-1})^4 \cdot \sqrt{-1} = \sqrt{-1}, \qquad \text{"} \quad i^5 = i,$$

and so on for the higher powers of $\sqrt{-1}$; any one of these powers, when simplified, will be found to have one or another of the four values: $\sqrt{-1}$, -1, $-\sqrt{-1}$, and 1.

EXERCISE CXIV

1. Define an imaginary number (cf. §§ 114 and 146).

2. Which of the following are imaginary numbers: $\sqrt{-3}$, $\sqrt[4]{-2}$, $\sqrt[3]{-6}$, $\sqrt{5}$, $\sqrt[6]{-a^2}$, $3\sqrt[4]{-7}$, $4a\sqrt{-\frac{1}{3}}$, and $\frac{2}{3} + \frac{1}{2}\sqrt{-5}$?

3. Is $\sqrt{-x}$ imaginary when x represents a positive number? when x represents a negative number? Answer the same questions for $\sqrt[3]{-x}$.

4. Reduce to the standard form (§ 164, Remark): $\sqrt{-9}$; $\sqrt{-c^2}$; $\sqrt{-10}$; $\sqrt{-13}$; $\sqrt{-12}$; and $\sqrt{-25c^3}$.

5. Show by the method of proof suggested in § 150 that $\sqrt{-7} = \sqrt{7} \cdot \sqrt{-1}$.

6. Show that if $i = \sqrt{-1}$, then

$$\begin{array}{lll} i^2 = -1, & i^6 = -1, & i^{10} = ? \\ i^3 = -i, & i^7 = -i, & i^{11} = ? \\ i^4 = 1, & i^8 = 1, & i^{12} = ? \\ i^5 = i, & i^9 = i, & i^{13} = ? \end{array}$$

7. Since any *even* number may be written in the form $2n$, where n is an integer, and since $a^{2n} = (a^2)^n$, show that every *even* power of i is real.

8. When n is even, does i^{2n} equal 1 or -1? Why? Answer the same questions if n is odd.

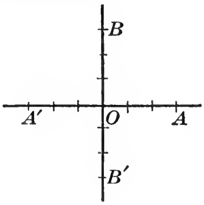
9. Give the values of the following even powers of i: $i^8\,[=(i^2)^4]$; i^{20}; i^{32}; i^{18}; i^{64}; i^{100}; i^6; i^{90}.

10. Show that every odd power of i is either i or $-i$ (cf. Ex. 7).

11. Find by inspection the value of the following odd powers of i: i^7; i^{15}; i^9; i^{31}; i^{19}; i^{25}.

12. Distinguish between pure and complex imaginary numbers, and give three examples of each.

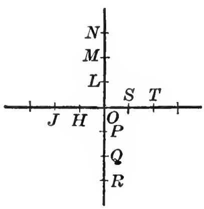
166. Graphical representation of imaginary numbers. Since opposite numbers, such as $+3$ and -3, are represented graphically by opposite distances, such as OA and OA'; and since multiplying $+3$ by -1 gives -3; therefore we may regard the multiplier -1 as an operator which rotates OA through 180° about O, into the position OA'.



Again, since $i \cdot i = -1$, therefore i may be regarded as an operator which when applied *twice* in succession rotates OA through 180°; hence using i as a multiplier *once* (instead of twice) should rotate OA through 90° into the position OB.

In other words: $3\sqrt{-1}$ (*i.e.*, $OA \cdot i$) may be represented graphically by OB. Similarly, any pure imaginary number whatever may be laid off on the line $B'OB$, *above* the origin O if the number is *positive*, *below* the origin if it is *negative*.

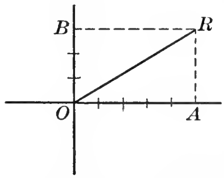
E.g., if each division on the lines JT and RN in the figure represents a unit, then $OS = 1$, $OJ = -2$, $OL = \sqrt{-1}$, $ON = 3\sqrt{-1}$, $OQ = -2\sqrt{-1}$, etc.



The lines JT and RN are often called the **axis of real numbers** and the **axis of imaginaries**, respectively.

167. Graphical representation of complex numbers. A complex number, such as $5 + 3\sqrt{-1}$, may be graphically represented as follows: lay off OA, 5 units on the axis of real numbers, and OB, 3 units on the axis of imaginaries, then complete the parallelogram $AOBR$, and draw its diagonal OR; this diagonal is a graphical repre-

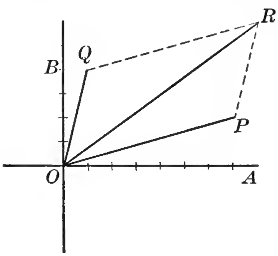
sentation of the complex number $5+3\sqrt{-1}$, *i.e.*, of the *sum* of 5 and $3\sqrt{-1}$.



Moreover, it is evident that any complex number whatever may be graphically represented by the above method.

Note. The appropriateness of calling OR the *sum* of OA and OB will be evident to pupils who have an elementary knowledge of physics. Thus, if two forces, represented in amount and direction by OA and OB, respectively, act simultaneously on a body at O, the result is the same as though a single force represented in amount and direction by OR were acting on this body; *i.e.*, the *sum* of the *forces* OA and OB is the *force* OR.

168. Graphical representation of the sum of complex numbers; also of their difference. The sum of two complex numbers, such as $7+2\sqrt{-1}$ and $1+4\sqrt{-1}$, may be graphically represented as follows: let OP and OQ be the graphical representations of $7+2\sqrt{-1}$ and $1+4\sqrt{-1}$, respectively (§ 167); complete the parallelogram $POQR$, and draw its diagonal OR; then OR is the graphical representation of the sum of $7+2\sqrt{-1}$ and $1+4\sqrt{-1}$ (cf. § 167, Note).



Obviously the sum of any two complex numbers may be represented by this method.

Again, to find the difference of two complex numbers graphically, we have only to reverse the sign of the subtrahend, and proceed as in addition.

EXERCISE CXV

Represent graphically the following imaginary numbers:

1. $2\sqrt{-1}$.	**3.** $-7i$.	**5.** $2.4i$.	**7.** $\sqrt{-5}$.
2. $-5\sqrt{-1}$.	**4.** $\frac{2}{3}i$.	**6.** $\sqrt{-9}$.	**8.** $\sqrt{-\frac{1}{4}}$.

Perform the following additions and subtractions graphically:

9. $4+2$ [cf. § 4].

10. $4i+2i$.

11. $4i-2i$.

12. $7-5$.

13. $3-7$.

14. $3i-10i$.

15. $3-8-6$.

16. $\sqrt{-25}+2\sqrt{-36}-\sqrt{-49}$.

Represent by drawing each of the following complex numbers:

17. $2+4\sqrt{-1}$.

18. $4+2\sqrt{-1}$.

19. $2-5\sqrt{-1}$.

20. $-7+i$.

21. $4+3i$.

22. $-6-6i$.

23. $3+\sqrt{-36}$.

24. $-2+\sqrt{-9}$.

25. $\sqrt{-\frac{1}{4}}+6$.

26. $\sqrt{-2}-7$.

27. $10+\sqrt{-18}$.

28. $-\frac{1}{2}-\sqrt{-8}$.

Perform the following indicated operations graphically:

29. $(1+4i)+(4+2i)$.

30. $(3+2i)+(5+3i)$.

31. $(2-5i)+(8+i)$.

32. $(3+\sqrt{-4})+(-2+\sqrt{-9})$.

33. $(5-\sqrt{-9})+(-3+\sqrt{-25})$.

34. $-2i+(-3+\sqrt{-7})$.

35. $(4+3i)-(2+i)$.

36. $(8-2i)-(5-3i)$.

37. $(10+\sqrt{-9})-(-2+\sqrt{-16})$.

38. $(\frac{1}{2}-\sqrt{-8})-(7-\sqrt{-2})$.

169. Fundamental operations with pure and complex imaginary numbers. If complex numbers are first *reduced to the standard form* $a+b\sqrt{-1}$, and if we are careful to remember that $\sqrt{-1}\cdot\sqrt{-1}=-1$ and not $+1$, then the operations of addition, subtraction, etc., with these numbers may be performed exactly as are the corresponding operations with real numbers. The following examples will illustrate these operations.

Ex. 1. Find the sum of $2+\sqrt{-9}$, $8-\sqrt{-4}$, and $-\sqrt{-25}$.

SOLUTION. Since

$$\begin{aligned} 2+\sqrt{-9} &= 2+3\sqrt{-1}, \\ 8-\sqrt{-4} &= 8-2\sqrt{-1}, \end{aligned}$$

and

$$-\sqrt{-25} = \quad -5\sqrt{-1},$$

therefore the sum is

$$10-4\sqrt{-1},\ i.e.,\ 10-\sqrt{-16}.$$

Ex. 2. Multiply $5\sqrt{-2}$ by $4\sqrt{-7}$.

SOLUTION. $\qquad 5\sqrt{-2} = 5\sqrt{2}\cdot\sqrt{-1},$

and $\qquad 4\sqrt{-7} = 4\sqrt{7}\cdot\sqrt{-1},$

hence the product is $\qquad 20\sqrt{2}\cdot\sqrt{7}\;(\sqrt{-1})^2$, *i.e.*, $-20\sqrt{14}$.

NOTE. Observe that this product is *not* $+20\sqrt{14}$, as it would be if the factors were *real* numbers. Beginners should be especially careful to guard against errors in the *sign* of a product of imaginary numbers.

Ex. 3. Multiply $3+\sqrt{-5}$ by $2-\sqrt{-3}$.

SOLUTION. Writing these imaginary numbers in terms of the imaginary unit, the work may be arranged thus:

$$\begin{array}{l}
3+\sqrt{5}\cdot\sqrt{-1} \\
2-\sqrt{3}\cdot\sqrt{-1} \\
\hline
6+2\sqrt{5}\cdot\sqrt{-1} \\
\quad -3\sqrt{3}\cdot\sqrt{-1}-\sqrt{15}(\sqrt{-1})^2 \\
\hline
6+(2\sqrt{5}-3\sqrt{3})\cdot\sqrt{-1}+\sqrt{15}.
\end{array}$$

Ex. 4. Divide $12+\sqrt{-25}$ by $3-\sqrt{-4}$.

SOLUTION. Such divisions are easily performed by first multiplying both dividend and divisor by the conjugate of the divisor, thus:

$$\frac{12+\sqrt{-25}}{3-\sqrt{-4}} = \frac{12+5\sqrt{-1}}{3-2\sqrt{-1}} = \frac{(12+5\sqrt{-1})(3+2\sqrt{-1})}{(3-2\sqrt{-1})(3+2\sqrt{-1})}$$

$$= \frac{36+39\sqrt{-1}+10(\sqrt{-1})^2}{9-4(\sqrt{-1})^2}$$

$$= \frac{26+39\sqrt{-1}}{9+4} = 2+3\sqrt{-1}.$$

170. Important property of complex numbers. By a method altogether like that used in § 161 it may be shown that if $a+b\sqrt{-1} = c+d\sqrt{-1}$, then $a=c$ and $b=d$. Moreover, this fact may be used, as in § 162, to extract the square root of any complex number (cf. *El. Alg.* §§ 151, 182).

EXERCISE CXVI

5. Add $3+5i$ and $7+\sqrt{-4}$ as in § 169; then add these numbers graphically, and check your work by comparing results.

Simplify Exs. 6–13 below, and check as teacher directs:

6. $7-6i+2+3i$.

7. $(3+2i)+(3-2i)$.

8. $(3+2i)-(3-2i)$.

9. $(-4-\sqrt{-8})+(-4-\sqrt{-8})$.

10. $\sqrt{-4}+4\sqrt{-9}+\sqrt{-25}$.

11. $3+\sqrt{-16}+\sqrt{-4}-5-\sqrt{-9}$.

12. $3+\sqrt{-36}-(1+2\sqrt{-25})+3\sqrt{-16}$.

13. $\sqrt{-49}+5\sqrt{-4}-(6+2\sqrt{-9})$.

Simplify each of the following expressions (cf. § 152):

14. $\sqrt{-\frac{1}{2}}-(2\sqrt{-\frac{2}{3}}+5-3\sqrt{-24})+3\sqrt{-18}$.

15. $\sqrt{-16a^2x^2}+\sqrt{1-5}+2\sqrt{5-30}-\sqrt{-9a^2x^2}+\sqrt{-a^2x^2}$.

16. $x\sqrt{-4}+\sqrt{-x^2-2x-1}-\sqrt{-32}$.

17. Solve the equation $x^3-1=0$ (cf. § 72) and find the sum of the roots; check your addition graphically. Similarly find the sum of the 3 cube roots of 8; of the 3 cube roots of -27.

Find the product of:

18. $3\sqrt{-6}$ by $5\sqrt{-12}$.

19. $5\sqrt{-8}$ by $2\sqrt{-6}$.

20. $2\sqrt{-4}$ by $\sqrt{-4a^2x^3}$.

21. $-i-6i^5+i^{45}$ by i^5.

22. $\sqrt{-6}+\sqrt{-3}$ by $\sqrt{-6}-\sqrt{-3}$.

23. $3+2\sqrt{-9}$ by $5-4\sqrt{-1}$.

24. $\sqrt{-50}-2\sqrt{-12}$ by $\sqrt{-8}-5\sqrt{-3}$.

25. $3i^2-4i^7$ by $2i^3-3i^{12}$.

26. Show that the sum, and also the product, of $a+bi$ and $a-bi$ is real (a and b being any real numbers). Show that the same is true also for $\sqrt{-4}-3$ and $-\sqrt{-4}-3$.

27. Show that both the sum and also the product of *any* two conjugate complex numbers is real.

28. Multiply $\sqrt{-a}+\sqrt{-b}+\sqrt{-c}$ by $\sqrt{-a}-\sqrt{-b}+\sqrt{-c}$.

29. $(1+\sqrt{-5})^2=?$ **30.** $(2-3i)^3=?$ **31.** $(2a-3x\sqrt{-1})^2=?$

32. Find the product of $a\sqrt{-b}+b\sqrt{-a}$, $a\sqrt{-a}+b\sqrt{-b}$, and $b\sqrt{-b}-a\sqrt{-a}$.

33. Show that $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$ and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$ are conjugates of each other, and also that each is the square of the other.

34. Reduce $\dfrac{2-3\sqrt{-1}}{3-\sqrt{-4}}+\dfrac{2+3i}{3+2i}$ to its simplest form.

35. Simplify each of the following indicated quotients:

$$\frac{\sqrt{-15}}{\sqrt{-3}};\ \frac{\sqrt{-24}}{\sqrt{-4}};\ \frac{\sqrt{-c}}{\sqrt{-d}};\ \frac{\sqrt{c}}{\sqrt{-d}};\ \frac{\sqrt{-c}}{\sqrt{d}};\ \frac{\sqrt{84}}{2\sqrt{-7}};\ \frac{\sqrt{-6}+2\sqrt{-8}}{\sqrt{-2}}.$$

36. Show that $\dfrac{a+b\sqrt{-1}}{c+d\sqrt{-1}}=\dfrac{ac+bd+(bc-ad)\sqrt{-1}}{c^2+d^2}$.

Perform the following indicated divisions (cf. Ex. 4), and check your work by multiplying the quotient by the divisor:

37. $\dfrac{4}{1+i}$.

38. $\dfrac{2}{i^4+i^8}$.

39. $\dfrac{2-\sqrt{-3}}{3+\sqrt{-2}}$.

40. $\dfrac{5+\sqrt{-4}}{5-2i}$.

41. $\dfrac{\sqrt{2x}-3ai}{\sqrt{2x}+2bi}$.

42. $\dfrac{\sqrt{a}-i\sqrt{b}}{i\sqrt{b}+\sqrt{a}}$.

43. If a and b are positive and unequal numbers, show that $\sqrt{-a}\pm\sqrt{-b}$ cannot equal a real number (cf. § 161).

44. Show that if $x+\sqrt{-y}=a+\sqrt{-b}$, wherein y and b are positive numbers, then $x=a$ and $y=b$.

45. Find the square root of $5-12\sqrt{-1}$.

HINT. Let $\sqrt{x}-\sqrt{y}\cdot\sqrt{-1}=\sqrt{5-12\sqrt{-1}}$ (cf. § 162).

Find the square root of:

46. $10-6\sqrt{-1}$.

47. $6\sqrt{-2}-17$.

48. $3+2\sqrt{-10}$.

49. $5\frac{1}{8}-3\frac{3}{4}\sqrt{-2}$.

CHAPTER XVI

THEORY OF EXPONENTS

ZERO, NEGATIVE, AND FRACTIONAL EXPONENTS

171. Introductory. (i) As originally defined (§ 9) an exponent is necessarily a positive integer, and it is in this sense only that we have thus far used it. Under this restriction we have established the following exponent laws (§ 110), wherein a is any real number except 0 :

$$\text{I} \qquad a^m \cdot a^n = a^{m+n},$$

$$\text{II} \qquad (a^m)^n = a^{mn},$$

$$\text{III} \qquad (ab)^n = a^n \cdot b^n,$$

$$\text{IV} \qquad a^m : a^n = a^{m-n}.$$

(ii) We now propose to *extend the meaning* of an exponent so as to include such symbols as a^0, a^{-5}, and $a^{\frac{2}{3}}$, along with our former exponent expressions.

In *extending* the meaning of any symbol already in use, however, the extended meaning should be such as not to disturb any rules of operation already established for the symbol in question. Hence, we shall admit such symbols as a^0, a^{-5}, etc., into our algebraic notation if, and only if, we can *assign* to each of these symbols a meaning consistent with the above exponent laws.

172. Meaning of such symbols as a^0, a^{-5}, and $a^{\frac{2}{3}}$. (1) If, following the plan given § 171 (ii), we let $n = 0$ in law I, § 171, we obtain

$$a^m \cdot a^0 = a^m, \qquad [\text{since } a^{m+0} = a^m$$

hence $$a^0 = 1\ ;$$

i.e., if law I is to admit the symbol a^0, then this symbol *must* have the value 1.

(2) Again, if in law I, § 171, we let $m = 5$ and $n = -5$, we obtain $$a^5 \cdot a^{-5} = a^0 = 1, \qquad [\text{since } a^{5-5} = a^0$$

whence $$a^{-5} = \frac{1}{a^5};$$

i.e., if law I is to admit the symbol a^{-5}, then this symbol must have the same meaning as $\frac{1}{a^5}$.

(3) And finally, if law I, § 171, is to remain valid for such symbols as $a^{\frac{2}{3}}$, then

$$a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^2, \qquad [\text{since } a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^2$$

and therefore $$a^{\frac{2}{3}} = \sqrt[3]{a^2}, \qquad [\S\ 113$$

i.e., if law I is to admit the symbol $a^{\frac{2}{3}}$, then this symbol must have the same meaning as $\sqrt[3]{a^2}$.

173 Definitions of a^0, a^{-k}, and $a^{\frac{p}{r}}$. Using as a basis the special cases considered in § 172, we shall now *define* the symbols a^0, a^{-k}, and $a^{\frac{p}{r}}$ as follows:

(1) $$a^0 = 1,$$

(2) $$a^{-k} = \frac{1}{a^k},$$

and (3) $$a^{\frac{p}{r}} = \sqrt[r]{a^p};$$

wherein a represents any number whatever, and k, p, and r are positive integers.

Moreover, these definitions — since they are based upon *one* exponent law only — must be regarded as *tentative* until they are shown (§§ 174, 175) to satisfy *all* of the exponent laws.

EXERCISE CXVII

Assuming the validity of § 173 (1) and (2), show that:

1. $3\,c^{-2} = 3 \cdot \frac{1}{c^2} = \frac{3}{c^2}.$

2. $x^0 y^{-5} = \frac{1}{y^5}.$

3. $a^0 b^0 c = c.$

4. $\frac{xy^{-1}}{z^{-1}} = \frac{xz}{y}.$

Reduce the following to equivalent expressions free from zero and negative exponents:

5. x^0.	**9.** $5\,k^{-2}$.	**13.** $8\,x^0 \div y^{-4}$.
6. x^{-3}.	**10.** $\frac{1}{2}\,k^{-6}$.	**14.** $y^2 \div y^{-7}$.
7. $6\,a^0$.	**11.** $7 \div r^{-3}$.	**15.** $y^2 \cdot y^{-7}$.
8. $-4 \div a^0$.	**12.** $8\,x^0 y^{-4}$.	**16.** $3\,x^{-2}y^{-2} \div xy^{-3}$.

Assuming the validity of § 173 (3), translate the following into equivalent radical expressions:

17. $a^{\frac{3}{5}}$.	**20.** $t^{\frac{1}{2}}$.	**23.** $(\frac{1}{2}\,x^2z^3)^{\frac{2}{3}}$.
18. $a^{\frac{5}{3}}$.	**21.** $(ax)^{\frac{m}{n}}$.	**24.** $(cd^{-2})^{\frac{2}{5}}$.
19 $x^{\frac{4}{7}}$.	**22.** $(4\,c^2d)^{\frac{1}{4}}$.	**25.** $\left(\frac{m^{-2}}{4}\right)^{\frac{3}{2}}$.

Write in the fractional exponent notation:

26. $\sqrt[4]{x^3}$.	**30.** $\sqrt[4]{a^2x^3}$.	**34.** $\sqrt[9]{-8\,mx^8}$.
27. $\sqrt[3]{m}$.	**31.** $\sqrt[10]{a^2x^5}$.	**35.** $b\sqrt[4]{16\,b^3}$.
28. $\sqrt{6}$.	**32.** $\sqrt[6]{3\,c^2d^3}$.	**36.** $2\sqrt{x^{-1}y^{-1}}$.
29. $\sqrt[5]{24}$.	**33.** $\sqrt[6]{\frac{2^3}{e^4d^{10}}}$.	**37.** $ac\sqrt[4]{a^0c^{-3}}$.

38. Find the numerical value of: 5^0; 3^{-3}; $4^{\frac{1}{2}}$; $9^{\frac{1}{2}}$; $(\frac{1}{8})^{\frac{1}{3}}$; $4^{-2} \cdot 2^3$.

39. Show that if law II, § 171, is to admit the symbol a^0, then a^0 must equal 1 (cf. § 172).

40. Show that if law IV, § 171, is to admit the symbol a^{-4}, then a^{-4} must equal $\frac{1}{a^4}$.

41. Show that if law II, § 171, is to hold for $a^{\frac{2}{3}}$, then $a^{\frac{2}{3}}$ must equal $\sqrt[3]{a^2}$.

174.* The symbols a^0 and a^{-k} obey all the exponent laws. That a^0 and a^{-k}, as defined in § 173, satisfy *all* the exponent laws may be shown by assigning zero and negative integral

* The *proofs* given in §§ 174 and 175 may, if the teacher prefers, be omitted until the subject is reviewed.

values to m and n, both separately and together, in the equations which express those laws.

Thus, if we let $m=0$ in law II (n remaining a positive integer), we obtain $(a^0)^n = a^0$, [since $a^{0\cdot n}=a^0$

i.e., $1^n = 1$,

which is correct; hence $a^0=1$ is consistent with law II.

Again, let $m=0$ in law IV (n remaining a positive integer), and we obtain $a^0 : a^n = a^{-n}$, [since $a^{0-n}=a^{-n}$

i.e., $1 : a^n = a^{-n}$,

which is a correct equation [§ 173 (2)]; hence $a^0=1$ and $a^{-n}=\frac{1}{a^n}$ are consistent with law IV.

Once more, let $m=-r$ and $n=-s$ (where r and s are positive integers) in law II, and we obtain

$$(a^{-r})^{-s} = a^{-r\cdot -s} = a^{rs};$$

but this is consistent with § 173 (2), for

$$(a^{-r})^{-s} = \left(\frac{1}{a^r}\right)^{-s} = \frac{1}{\left(\frac{1}{a^r}\right)^s} \qquad \text{[§ 173 (2)}$$

$$= \frac{1}{\frac{1}{a^{rs}}} = a^{rs}. \qquad \text{[§§ 109, 92}$$

Moreover, if we similarly test the remaining combinations of positive and negative integral and zero values of m and n, in the four exponent laws, we find that definitions (1) and (2) of § 173, and the exponent laws of § 171, are entirely consistent. Hence we need no longer regard the definitions of a^0 and a^{-k} as tentative (cf. § 173, last part).

The testing of some or all of these remaining cases may be assigned as an exercise to the pupil.

175. The symbol $a^{\frac{p}{r}}$ satisfies all the exponent laws. That $a^{\frac{p}{r}} = \sqrt[r]{a^p}$ is consistent with the exponent laws may be shown as follows:

Let p, r, s, and t represent any positive integers, then

$$a^{\frac{p}{r}} \cdot a^{\frac{s}{t}} = \sqrt[r]{a^p} \cdot \sqrt[t]{a^s}$$

$$= \sqrt[rt]{a^{pt}} \cdot \sqrt[rt]{a^{rs}} = \sqrt[rt]{a^{pt} \cdot a^{rs}} \qquad [\S\S\ 154,\ 149\ (i)$$

$$= \sqrt[rt]{a^{pt+rs}} = a^{\frac{pt+rs}{rt}} = a^{\frac{p}{r}+\frac{s}{t}};$$

i.e., law I holds good for such symbols as $a^{\frac{p}{r}}$.

Similarly, $$\left(a^{\frac{p}{r}}\right)^{\frac{s}{t}} = \left(\sqrt[r]{a^p}\right)^{\frac{s}{t}} = \sqrt[t]{\left(\sqrt[r]{a^p}\right)^s}$$

$$= \sqrt[tr]{a^{ps}} \qquad [\S\ 149\ (iii)$$

$$= a^{\frac{ps}{tr}} = a^{\frac{p}{r}\cdot\frac{s}{t}};$$

i.e., law II holds good for such symbols as $a^{\frac{p}{r}}$.

Again, $$(ab)^{\frac{p}{r}} = \sqrt[r]{(ab)^p} = \sqrt[r]{a^p \cdot b^p}$$

$$= \sqrt[r]{a^p} \cdot \sqrt[r]{b^p} \qquad [\S\ 149\ (i)$$

$$= a^{\frac{p}{r}} b^{\frac{p}{r}};$$

i.e., law III holds good for such symbols as $a^{\frac{p}{r}}$.

The proof for law IV, being closely similar to that for law I, is left as an exercise for the pupil.

Observe that the above proofs remain valid:

(1) if r (or t) takes the value 1, in which case $\frac{p}{r}$ $\left(\text{or } \frac{s}{t}\right)$ becomes an integer.

(2) if p (or s) is negative or zero (cf. § 174), in which case $\frac{p}{r}$ $\left(\text{or } \frac{s}{t}\right)$ becomes a negative fraction or integer, or zero.

Therefore these proofs, taken in connection with those previously given, include all possible combinations of positive and negative, integral, fractional, and zero values of m and n, in the exponent laws of § 171. We need, therefore, no longer regard the definition of $a^{\frac{p}{r}}$ [§ 173 (3)] as tentative.

NOTE. Observe also that $a^{\frac{p}{r}}$ represents the *principal rth* root of a^p, since that is the meaning of the symbol $\sqrt[r]{a^p}$ (§ 150).

EXERCISE CXVIII

1. Does a^0 equal x^0 even when a and x are unequal? Explain.

2. By means of § 173 (1) show that $a^m \div a^n = a^{m-n}$ when $n = 0$ (cf. § 171, IV).

3. By means of § 173 (2) show that $(a^2)^{-3} = a^{-6}$ and that $a^{-5} \div a^{-3} = a^{-2}$ (cf. § 171, II and IV).

4. Show that $\dfrac{6\,a^2x^{-3}}{y^{-4}} = \dfrac{6\,a^2y^4}{x^3}$ and that $\dfrac{3^{-2}n^3x^{-1}}{2^4x^4y^{-3}} = \dfrac{2^{-4}x^{-4}y^3}{3^2n^{-3}x}$.

5. By proceeding as in Ex. 4 show that, by changing the sign of the exponent, a *factor* may be transferred from the numerator to the denominator of a fraction, and *vice versa*.

6. Is $\dfrac{a^2 + b^{-3}c^4}{4\,x}$ equal to $\dfrac{a^2 + c^4}{4\,b^3x}$? Explain. Observe carefully that a *factor* but not a *part* may be transferred as in Ex. 5.

Free the following expressions from negative exponents, and explain your work in each case:

7. $\dfrac{a^2}{b^{-2}}$.

8. $\dfrac{c^3d^{-1}}{2\,d^4}$.

9. $\dfrac{a^{-1}x^{-3}}{b^{-1}x^{-1}}$.

10. $\dfrac{3 \cdot 2^{-2}}{8^{-2}}$.

11. $\dfrac{5^2 \cdot 12^{-1}}{10^{-1} \cdot 3^4}$.

12. $a^{-1} + 2$.

13. $\dfrac{x^{-2} + y^{-1}}{5}$.

14. $\dfrac{m^{-3} - n^{-3}}{m^{-3} \cdot n^{-3}}$.

15. $\dfrac{s^{-4}(s+1)^{-2}}{(s^2-1)^{-1}}$.

16. $\dfrac{9^{-2}x^a}{6\,x^{-b}}$.

17. $\dfrac{9^{-2}x^a}{(-6)^{-1}(x^2y)^{-c}}$.

18. $\dfrac{3\,r^{-2} - 4\,s}{16\,s^2 - 9\,r^{-4}}$.

19. Is $\dfrac{3\,a}{x^2}$ equal to $3\,ax^{-2}$? Why?

20. As in Ex. 19, write the following fractions in integral form:

$\dfrac{a^2x}{b^2y}$; $\dfrac{4\,a^{-3}c^{-5}}{ac}$; $\dfrac{4^{-1}a^{-3}c^5}{5a^{-1}d^{-1}}$; $\dfrac{m^{-1} - 3\,n^{-4}}{m^{-2}n}$.

21. What is the difference in meaning between $m^{\frac{2}{3}}$ and $m^{\frac{3}{2}}$? between $m^{\frac{2}{3}}$ and $m^{-\frac{2}{3}}$, *i.e.*, between $m^{\frac{2}{3}}$ and $m^{\frac{-2}{3}}$?

Write Exs. 22–26 below as radical expressions, and write Exs. 27–30 with positive fractional exponents:

22. $a^{-\frac{3}{10}}b^{\frac{7}{10}}$.

23. $2^{\frac{5}{6}}d^{\frac{5}{6}}e^{-\frac{4}{6}}$.

24. $\left(\frac{2x}{3}\right)^{\frac{2}{3}}$.

25. $5^{-\frac{1}{a}}y^{\frac{a+c}{a}}$.

26. $a^{\frac{1}{m}} - b^{\frac{n}{8}}$.

27. $\sqrt[3]{-\frac{1}{8}c^{-6}d^{-2}e}$.

28. $\sqrt[q]{r^{-p}s^{2p}}$.

29. $\sqrt[5]{-32\,r^{-4}s^{10}}$.

30. $\sqrt[k]{(c+d)^e} \div \sqrt[e]{(c+d)^k}$.

Find the numerical value of:

31. $4^{\frac{1}{2}}$.

32. 4^{-2}.

33. $9^{\frac{3}{2}}$.

34. $8^{-\frac{2}{3}} \cdot 25^{\frac{3}{2}}$.

35. $6^0 \div 2^{-4}$.

36. $(.09)^{\frac{5}{2}}$.

37. $(256)^{\frac{3}{4}}$.

38. $(\frac{125}{216})^{-\frac{2}{3}}$

39. $\left(\frac{27\,x^0}{.008}\right)^{\frac{2}{3}}$.

40. $(32^{-2})^{\frac{1}{5}} \div (32^{\frac{1}{5}})^{-2}$.

41. $(-\frac{1}{243})^{\frac{1}{5}} \cdot (169)^{\frac{1}{2}}$.

42. $(64)^{-\frac{5}{6}} \cdot (16^{-\frac{1}{2}})^3$.

43. $\left(\frac{16}{625}\right)^{\frac{3}{4}} \cdot \left(\frac{4}{5}\right)^{-1} \div \left(\frac{343}{8}\right)^{-\frac{2}{3}} - 18 \cdot 10^{-2}$.

176. Operations with negative, zero, and fractional exponents. From the definitions of negative, zero, and fractional exponents (§§ 173–175) it follows that in all operations with such symbols as a^0, a^{-5}, and $a^{\frac{2}{3}}$, their exponents obey the exponent laws of § 171; that is, *these exponents behave just as though they were positive integers.*

When working with fractional-exponent expressions it is frequently necessary to change the exponents to higher or lower terms; this does not change the value of such an expression;

for since $$\sqrt[r]{a^p} = \sqrt[rt]{a^{pt}}, \qquad [§ 151 (iv)$$

therefore $$a^{\frac{p}{r}} = a^{\frac{pt}{rt}}.$$

Operations with radicals may often be greatly simplified by first converting the various radical expressions into their fractional-exponent equivalents (cf. Exs. 29, 41, and 57 in the following exercise).

EXERCISE CXIX

1. Simplify: $\sqrt{2}\cdot\sqrt[3]{4}\cdot 2^{\frac{1}{3}}\cdot 2^{-\frac{5}{6}}$.

Solution. $\sqrt{2}\cdot\sqrt[3]{4}\cdot 2^{\frac{1}{3}}\cdot 2^{-\frac{5}{6}} = 2^{\frac{1}{2}}\cdot 2^{\frac{2}{3}}\cdot 2^{\frac{1}{3}}\cdot 2^{-\frac{5}{6}}$ [§ 173 (3)

$= 2^{\frac{1}{2}+\frac{2}{3}+\frac{1}{3}-\frac{5}{6}} = 2^{\frac{2}{3}} = \sqrt[3]{4}$. [§ 171, I

2. Simplify: $(ab)^{\frac{1}{3}}\cdot(b^2c)^{\frac{1}{3}}$.

Solution. $(ab)^{\frac{1}{3}}\cdot(b^2c)^{\frac{1}{3}} = (ab^3c)^{\frac{1}{3}} = a^{\frac{1}{3}}bc^{\frac{1}{3}}$. [§ 171, III, II

3. Simplify: $\left(\dfrac{b\sqrt[3]{a^2}}{a\sqrt{b}}\right)^{-4}$.

Solution. $\left(\dfrac{b\sqrt[3]{a^2}}{a\sqrt{b}}\right)^{-4} = \left(\dfrac{ba^{\frac{2}{3}}}{ab^{\frac{1}{2}}}\right)^{-4} = \left(\dfrac{b^{\frac{1}{2}}}{a^{\frac{1}{3}}}\right)^{-4} = \dfrac{b^{-2}}{a^{-\frac{4}{3}}}$ [§ 171, II

$= \dfrac{a^{\frac{4}{3}}}{b^2} = \dfrac{a\sqrt[3]{a}}{b^2}$.

Perform the following indicated operations, and express your results in their simplest form:

4. $8^{\frac{2}{3}}\cdot 8^{\frac{5}{6}}\cdot 8^{\frac{1}{2}}$.

5. $a^{\frac{2}{5}}\cdot a^{\frac{1}{2}}\cdot a^{-\frac{3}{4}}$.

6. $2^{\frac{1}{2}}\cdot 8^{-\frac{1}{2}}\cdot 27^{\frac{1}{2}}$.

7. $(y)^{\frac{3}{4}}\cdot\left(\dfrac{1}{x^3}\right)^{\frac{3}{4}}\cdot\left(\dfrac{y}{x}\right)^{\frac{3}{4}}$.

8. $2x^{\frac{3}{8}} \div 4x^{\frac{1}{4}}$.

9. $\sqrt[3]{s^2}\cdot\sqrt[4]{s^3}$.

10. $s\sqrt[3]{s^{-2}} \div \sqrt[4]{s^{-3}}$.

11. $(\sqrt[6]{a^5}\cdot a^{-\frac{7}{12}})^4$.

12. $(36a^{-\frac{2}{3}}c^{\frac{1}{3}})^{-\frac{3}{2}}$.

13. By means of fractional exponents, reduce $\sqrt[3]{a^2}$ and $\sqrt{x^5}$ to equivalent radicals of the same order (cf. § 154).

Solution. The given radicals are, respectively, equivalent to $a^{\frac{2}{3}}$ and $x^{\frac{5}{2}}$, and these expressions are, respectively, equivalent to $a^{\frac{4}{6}}$ and $x^{\frac{15}{6}}$, *i.e.*, to $\sqrt[6]{a^4}$ and $\sqrt[6]{x^{15}}$, each of which is of order 6.

14. Solve Exs. 12–16, p. 245, by means of fractional exponents.

15. By means of fractional exponents, solve Exs. 19–22, p. 247, Exs. 10–14, p. 251, and Exs. 9–12, 25–29, p. 252.

By means of fractional exponents, simplify:

16. $\sqrt{x} \div 3\sqrt[3]{a^2}\cdot\sqrt[4]{x}$.

17. $(\sqrt[5]{m^3}\cdot\sqrt{n^{-4}})^{\frac{5}{6}}$.

18. $(\sqrt[4]{r^{-3}}\cdot\sqrt{s}\cdot\sqrt[8]{t})^{-a}$.

19. $(8^{\frac{5}{6}}\cdot 8^{-\frac{1}{2}})^{-2}$.

20. $\sqrt[3]{8^4} \div \dfrac{2}{\sqrt[3]{8^{-5}}}$.

21. $\sqrt[3]{4\,a^{-2}} \div \sqrt[3]{-8\,a^{-3}}$.

22. $(\sqrt[3]{s^2})^{-2} \cdot (3\,s^{-\frac{1}{5}}y^{-\frac{1}{3}})^2$.

23. $\sqrt[m]{x^{-1}} \cdot \sqrt[n]{x^{-2}}$.

24. $(\sqrt[n]{a^t} \div \sqrt[m]{a^{-s}})^{-2mn}$.

25. $\left(\dfrac{m^{\frac{3}{5}}n^{-2}}{27\,m^{\frac{3}{10}}n}\right)^{\frac{1}{3}}$.

26. $\dfrac{(a^{-n} + a^{\frac{m}{2}})\sqrt{a}}{3\sqrt[3]{a^{-2m}}}$.

27. $\dfrac{m^{\frac{1}{2}} - m^{-\frac{1}{3}} + \sqrt[3]{7\,m^2}}{m^{\frac{1}{3}}}$.

28. $\dfrac{\dfrac{v}{v^{\frac{5}{2}}} \cdot 25^{\frac{1}{2}}v^{-\frac{1}{2}}}{v^{-\frac{5}{2}} + \dfrac{5}{v^{\frac{1}{2}}}}$.

29. Find the product of $3\sqrt{a} - 5\sqrt[3]{y}$ by $2\sqrt{a} + \sqrt[3]{y}$.

Solution. Since $3\sqrt{a} - 5\sqrt[3]{y} = 3\,a^{\frac{1}{2}} - 5\,y^{\frac{1}{3}}$, and $2\sqrt{a} + \sqrt[3]{y} = 2\,a^{\frac{1}{2}} + y^{\frac{1}{3}}$, therefore this product becomes

$$\begin{array}{r} 3\,a^{\frac{1}{2}} - 5\,y^{\frac{1}{3}} \qquad\qquad \\ 2\,a^{\frac{1}{2}} + y^{\frac{1}{3}} \qquad\qquad \\ \hline 6\,a^{\frac{1}{2}+\frac{1}{2}} - 10\,a^{\frac{1}{2}}y^{\frac{1}{3}} \qquad\qquad \\ + 3\,a^{\frac{1}{2}}y^{\frac{1}{3}} - 5\,y^{\frac{1}{3}+\frac{1}{3}} \\ \hline 6\,a - 7\,a^{\frac{1}{2}}y^{\frac{1}{3}} - 5\,y^{\frac{2}{3}}. \end{array}$$

If it is desired, this product may, of course, be written in either of the following forms: $6\,a - 7\sqrt{a}\sqrt[3]{y} - 5\sqrt[3]{y^2}$, or $6\,a - 7\sqrt[6]{a^3y^2} - 5\sqrt[3]{y^2}$.

Perform the following multiplications:

30. $a^{\frac{1}{2}} + b^{\frac{1}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

31. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.

32. $m^{\frac{2}{5}} - m^{\frac{1}{5}}n^{\frac{1}{5}} + n^{\frac{2}{5}}$ by $m^{\frac{1}{5}} + n^{\frac{1}{5}}$.

33. $m^{\frac{2}{5}} - m^{\frac{1}{5}}n^{-\frac{1}{5}} + n^{-\frac{2}{5}}$ by $m^{\frac{1}{5}} + n^{-\frac{1}{5}}$.

34. $\frac{1}{8}\,x^{\frac{3}{2}} - \frac{1}{12}\,xy^{\frac{1}{2}} + \frac{1}{18}\,x^{\frac{1}{2}}y - \frac{1}{27}\,y^{\frac{3}{2}}$ by $\frac{1}{2}\,x^{\frac{1}{2}} + \frac{1}{3}\,y^{\frac{1}{2}}$.

35. $81\,x^{\frac{4}{5}} - 27\sqrt[5]{x^3}\sqrt[3]{y} + 9\sqrt[5]{x^2}\sqrt[3]{y^2} - 3\,y\sqrt[5]{x} + y^{\frac{4}{3}}$ by $3\sqrt[5]{x} + y^{\frac{1}{3}}$.

36. $\sqrt{a} - 4\sqrt[8]{a^3x} + 6\sqrt[4]{ax} - 4\sqrt[8]{ax^3} + \sqrt{x}$ by $\sqrt[4]{a} - 2\sqrt[8]{ax} + \sqrt[4]{x}$.

37. $m^{\frac{2}{3}} + m^{-\frac{2}{3}} - 2\,m^{\frac{1}{3}} + 4\,m^{-\frac{1}{3}}$ by $1 + 2\,m^{\frac{1}{3}} - \dfrac{1}{\sqrt[3]{m}}$.

38. $p^{-\frac{3}{2}} + q^{-1.6} - p^{-.75}q^{-\frac{4}{5}}$ by $p^{-.75} + q^{-\frac{4}{5}}$.

39. $1\frac{4}{5}\,n^{\frac{1}{2}}x\sqrt[5]{x} + 2\,n\sqrt{n} + \frac{2}{5}\,x^{1.8} + 6\,n\sqrt[5]{x^3}$ by $\sqrt{n} - 3\,x^{\frac{3}{5}} + \dfrac{5}{2\,a}\,x^{\frac{4}{5}}$.

40. $5\,a^{-3}x^{\frac{2}{3}} + 3\,a^{.2}b^nx^{-1} - b^{2-n}x^{\frac{1}{2}}$ by $x^{-p} - 3\,b^{n-\frac{1}{2}}x^{.5} + a^{\frac{7}{2}}$.

41. Divide $x^2 - y^3$ by $\sqrt[3]{x} + \sqrt{y}$.

SOLUTION. Since $\sqrt[3]{x} + \sqrt{y} = x^{\frac{1}{3}} + y^{\frac{1}{2}}$, this solution may be put into the following form:

$$\begin{array}{l|l}
x^2 - y^3 & x^{\frac{1}{3}} + y^{\frac{1}{2}} \\
\cline{2-2}
\underline{x^2 + x^{\frac{5}{3}}y^{\frac{1}{2}}} & x^{\frac{5}{3}} - x^{\frac{4}{3}}y^{\frac{1}{2}} + xy - x^{\frac{2}{3}}y^{\frac{3}{2}} + x^{\frac{1}{3}}y^2 - y^{\frac{5}{2}} \\
- x^{\frac{5}{3}}y^{\frac{1}{2}} - y^3 & \\
\underline{- x^{\frac{5}{3}}y^{\frac{1}{2}} - x^{\frac{4}{3}}y} & \\
\qquad x^{\frac{4}{3}}y - y^3 & \\
\qquad \underline{x^{\frac{4}{3}}y + xy^{\frac{3}{2}}} & \\
\qquad\qquad - xy^{\frac{3}{2}} - y^3 & \\
\qquad\qquad \underline{- xy^{\frac{3}{2}} - x^{\frac{2}{3}}y^2} & \\
\qquad\qquad\qquad x^{\frac{2}{3}}y^2 - y^3 & \\
\qquad\qquad\qquad \underline{x^{\frac{2}{3}}y^2 + x^{\frac{1}{3}}y^{\frac{5}{2}}} & \\
\qquad\qquad\qquad\qquad - x^{\frac{1}{3}}y^{\frac{5}{2}} - y^3 & \\
\qquad\qquad\qquad\qquad \underline{- x^{\frac{1}{3}}y^{\frac{5}{2}} - y^3} & \\
\qquad\qquad\qquad\qquad\qquad 0 &
\end{array}$$

The above quotient may also be written thus:

$$\sqrt[3]{x^5} - \sqrt[3]{x^4}\,\sqrt{y} + xy - \sqrt[3]{x^2}\,\sqrt{y^3} + \sqrt[3]{x}\cdot y^2 - \sqrt{y^5}.$$

NOTE. To appreciate one of the advantages of fractional exponents the student has only to perform the division in Ex. 41, using the radical notation, and compare his work with the above solution.

Perform the following divisions:

42. $a + x^2$ by $a^{\frac{1}{3}} + x^{\frac{2}{3}}$.

43. $m^{\frac{4}{5}} - n^{\frac{2}{3}}$ by $m^{\frac{1}{5}} - n^{\frac{1}{6}}$.

44. $x^{-1} + 3\,y^{-\frac{1}{2}} - 10\,xy^{-1}$ by $x^{-1}\sqrt{y} - 2$.

45. $a^{\frac{2}{5}} + 2\sqrt[5]{a}\,b^{-\frac{1}{2}} + \dfrac{1}{b}$ by $\sqrt[5]{a} + b^{-\frac{1}{2}}$.

46. $x^{\frac{5}{2}} + x^2\sqrt[3]{y} - x\sqrt{x}\,y^{\frac{2}{3}} - xy + \sqrt{x}\,y^{\frac{4}{3}} + y^{\frac{5}{3}}$ by $\sqrt{x} + \sqrt[3]{y}$.

Simplify the following expressions:

47. $\left(\dfrac{\sqrt{x} + \sqrt[3]{y}}{\sqrt{x} - \sqrt[3]{y}}\right)^2 \cdot \dfrac{\sqrt[3]{x} - \sqrt{y}}{\sqrt[3]{x} + \sqrt{y}}$.

49. $\dfrac{x - y}{\sqrt{x} - \sqrt{y}} - \dfrac{\sqrt{x^3} - y^{\frac{3}{2}}}{x - y}$.

48. $\dfrac{x^m + y^n}{x^{-m} + y^{-n}} \cdot \dfrac{x^n - y^m}{x^{-n} - y^{-m}}$.

50. $\dfrac{\sqrt{y}}{y + \sqrt{y} + 1} \div \dfrac{1}{y^{\frac{3}{2}} - 1}$.

51. $\dfrac{a}{\sqrt[3]{a}-1}-\dfrac{a^{\frac{2}{3}}}{\sqrt[3]{a}+1}-\dfrac{1}{a^{\frac{1}{3}}-1}+\dfrac{1}{a^{\frac{1}{3}}+1}$.

Write down, by inspection, the square root of each of the following expressions:

52. $1-2u^{\frac{1}{3}}+u^{\frac{2}{3}}$.

53. $x^{\frac{4}{5}}+4x^{\frac{2}{5}}+4$.

54. $p^{\frac{1}{2}}-4+4p^{-\frac{1}{2}}$.

55. $ax^{\frac{2}{3}}+2a^{\frac{5}{6}}x^{\frac{5}{6}}+a^{\frac{2}{3}}x$.

56. $m+n+p-2m^{\frac{1}{2}}n^{\frac{1}{2}}+2n^{\frac{1}{2}}p^{\frac{1}{2}}-2m^{\frac{1}{2}}p^{\frac{1}{2}}$.

57. Extract the square root of $\sqrt[5]{x^4}-2\sqrt[5]{x^3}+5\sqrt[5]{x^2}-4\sqrt[5]{x}+4$.

Solution. This expression written in the equivalent fractional-exponent form is $x^{\frac{4}{5}}-2x^{\frac{3}{5}}+5x^{\frac{2}{5}}-4x^{\frac{1}{5}}+4$, and in this form its square root may be extracted just as though it were a rational expression (cf. § 117); thus:

$$
\begin{array}{l|l|l}
 & x^{\frac{4}{5}}-2x^{\frac{3}{5}}+5x^{\frac{2}{5}}-4x^{\frac{1}{5}}+4 & x^{\frac{2}{5}}-x^{\frac{1}{5}}+2 \\
 & x^{\frac{4}{5}} & \\
\hline
2x^{\frac{2}{5}}-x^{\frac{1}{5}} & -2x^{\frac{3}{5}}+5x^{\frac{2}{5}} & \\
 & -2x^{\frac{3}{5}}+x^{\frac{2}{5}} & \\
\hline
2x^{\frac{2}{5}}-2x^{\frac{1}{5}}+2 & 4x^{\frac{2}{5}}-4x^{\frac{1}{5}}+4 & \\
 & 4x^{\frac{2}{5}}-4x^{\frac{1}{5}}+4 & \\
\hline
 & 0 &
\end{array}
$$

hence the required root is $x^{\frac{2}{5}}-x^{\frac{1}{5}}+2$, *i.e.*, $\sqrt[5]{x^2}-\sqrt[5]{x}+2$.

Extract the square root of each of the following expressions:

58. $x^2+2x^{\frac{3}{2}}+3x+4x^{\frac{1}{2}}+3+2x^{-\frac{1}{2}}+x^{-1}$.

59. $a^{\frac{2}{3}}-4a^{\frac{5}{6}}+4a+2a^{\frac{7}{6}}-4a^{\frac{4}{3}}+a^{\frac{5}{3}}$.

60. $n^{\frac{8}{5}}-2m^{-\frac{3}{5}}n^{\frac{11}{5}}+2m^{\frac{4}{5}}n^{\frac{4}{5}}+m^{-\frac{6}{5}}n^{\frac{14}{5}}-2m^{\frac{1}{5}}n^{\frac{7}{5}}+m^{\frac{8}{5}}$

Extract the cube root of the following expressions; write the results with all exponents positive, and then replace all fractional-exponent forms by radical signs (cf. footnote, p. 185):

61. $8+12x^{\frac{2}{3}}+6x^{\frac{4}{3}}+x^2$.

62. $8x^{-1}-12x^{-\frac{2}{3}}y+6x^{-\frac{1}{3}}y^2-y^3$.

63. $t^{-\frac{3}{2}}-6t^{-1}+15t^{-\frac{1}{2}}-20+15t^{\frac{1}{2}}-6t+t^{\frac{3}{2}}$.

64. $8a^3b^{-\frac{3}{2}}+9ab^{\frac{1}{2}}+13a^{\frac{3}{2}}+3a^{\frac{1}{2}}b+18a^2b^{-\frac{1}{2}}+b^{\frac{3}{2}}+12a^{\frac{5}{2}}b^{-1}$.

Solve the following equations:

65. $m^{\frac{1}{3}} = 4.$ **67.** $x^{-\frac{1}{2}} = 5.$ **69.** $x^{-\frac{3}{2}} = -27.$

66. $t^{\frac{3}{4}} = 8.$ **68.** $\frac{1}{4}y^{\frac{2}{3}} = 25.$ **70.** $\sqrt{m^{\frac{3}{2}}} = 3\sqrt{3}.$

71. $2r^{\frac{2}{3}} + 5r^{\frac{1}{3}} - 3 = 0.$ [HINT. Put $y = r^{\frac{1}{3}}$].

72. $x^{-\frac{2}{3}} + 5x^{-\frac{1}{3}} + 4 = 0.$

73. $(2k-3)^{-2} + 7(2k-3)^{-1} - 8 = 0.$

177. Rationalizing factors of binomial surds. Another advantage of the fractional-exponent notation is that it furnishes an easy method for finding the rationalizing factor of any binomial surd whatever, — only *quadratic* binomial surds were considered in § 157.

Ex. 1. Find the rationalizing factor of $x^{\frac{1}{3}} + y^{\frac{1}{2}}$.

SOLUTION. Since $(x^{\frac{1}{3}})^n - (y^{\frac{1}{2}})^n$ is exactly divisible by $x^{\frac{1}{3}} + y^{\frac{1}{2}}$ when n is any positive *even* integer [§ 66 (ii)], and since 6 is the smallest value of n for which both $(x^{\frac{1}{2}})^n$ and $(y^{\frac{1}{3}})^n$ are rational, therefore the rationalizing factor (cf. § 157) is

$$\frac{(x^{\frac{1}{3}})^6 - (y^{\frac{1}{2}})^6}{x^{\frac{1}{3}} + y^{\frac{1}{2}}} = \frac{x^2 - y^3}{x^{\frac{1}{3}} + y^{\frac{1}{2}}} = x^{\frac{5}{3}} - x^{\frac{4}{3}}y^{\frac{1}{2}} + xy - x^{\frac{2}{3}}y^{\frac{3}{2}} + x^{\frac{1}{3}}y^2 - y^{\frac{5}{2}}.$$

Ex. 2. Find the rationalizing factor of $x^{\frac{1}{5}} + y^{\frac{2}{3}}$.

SOLUTION. As in Ex. 1 [cf. § 66 (iv)], we find this factor to be

$$\frac{(x^{\frac{1}{5}})^{15} + (y^{\frac{2}{3}})^{15}}{x^{\frac{1}{5}} + y^{\frac{2}{3}}} = \frac{x^3 + y^{10}}{x^{\frac{1}{5}} + y^{\frac{2}{3}}} = x^{\frac{14}{5}} - x^{\frac{13}{5}}y^{\frac{2}{3}} + x^{\frac{12}{5}}y^{\frac{4}{3}} - x^{\frac{11}{5}}y^2 + \cdots.$$

EXERCISE CXX

Find the rationalizing factors of the following expressions:

3. $s^{\frac{1}{3}} + t^{\frac{1}{3}}.$ **7.** $3v^{\frac{1}{6}} - w^{\frac{2}{3}}.$ **11.** $a^{\frac{1}{2}}b^{\frac{2}{5}} + 3v^2.$

4. $s^{\frac{1}{3}} - t^{\frac{1}{3}}.$ **8.** $s^{-\frac{4}{5}} + t^{\frac{5}{3}}.$ **12.** $x^{-\frac{1}{3}} + 2y^{\frac{3}{4}}.$

5. $a^{\frac{2}{3}} - x^{\frac{1}{4}}.$ **9.** $2m^{\frac{1}{2}} - n^{\frac{1}{8}}.$ **13.** $x^{-\frac{3}{4}} - l^{\frac{1}{5}}.$

6. $m^{\frac{1}{2}} + n^{\frac{3}{8}}.$ **10.** $2x^{\frac{3}{2}} - 3y^{\frac{1}{3}}.$ **14.** $2r^{-\frac{1}{2}}s^{-\frac{1}{4}} - t^{-\frac{5}{6}}.$

CHAPTER XVII

QUADRATIC EQUATIONS

[Supplementary to Chapter XII]

178. Solution of quadratic equations by means of a formula. Since every quadratic equation in one unknown number may be reduced to the form $ax^2 + bx + c = 0$ (§ 122), and since the roots of this equation are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, *whatever the numbers represented by a, b, and c* (Ex. 3, p. 195), therefore the roots of any *particular* quadratic equation may be found by merely substituting for a, b, and c, in the roots of the above *general* equation, those values which these coefficients have in the particular equation under consideration.

E.g., since the roots of $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, therefore the roots of $3x^2 + 10x - 8 = 0$ (in which $a = 3$, $b = 10$, and $c = -8$) are

$$\frac{-10 \pm \sqrt{10^2 - 4 \cdot 3 \cdot (-8)}}{2 \cdot 3}, \text{ i.e., } \frac{-10 \pm 14}{6}, \text{ i.e., } \frac{2}{3} \text{ and } -4.$$

So, too, the roots of $6y^2 + 19y - 7 = 0$ are

$$\frac{-19 \pm \sqrt{19^2 - 4 \cdot 6(-7)}}{2 \cdot 6}, \text{ i.e., } \frac{1}{3} \text{ and } -\frac{7}{2};$$

and the roots of $x^2 - 3x + 5 = 0$ are $\frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1}$,

i.e., $\frac{3 \pm \sqrt{-11}}{2}$.

NOTE. While the student should, of course, be able to solve quadratic equations without the use of the formula, he is advised to commit this formula to memory, and henceforth to employ it freely; he will find this well worth his while, because roots of quadratic equations are so often required in mathematical investigations.

179. Character of the roots of $ax^2 + bx + c = 0$. Discriminant. As we have already seen (§ 126), the roots of the equation $ax^2 + bx + c = 0$ are $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Hence it follows that, if a, b, and c represent real and rational numbers, the roots can be imaginary or irrational only if $\sqrt{b^2 - 4ac}$ is imaginary or irrational; *i.e.*,

If $b^2 - 4ac$ is positive, the roots are real and unequal;
if $b^2 - 4ac = 0$, the roots are real and equal;
if $b^2 - 4ac$ is negative, both roots are imaginary;
and the roots are rational only when $b^2 - 4ac$ is an exact square.
(Let the pupil fully explain each case.)

The expression $b^2 - 4ac$, which determines the character of the roots, is usually called the **discriminant** of the equation.

Thus, without actually solving the equation $3x^2 - 5x - 1 = 0$, we know that its roots are real, irrational, and unequal because, for this equation, $\sqrt{b^2 - 4ac} = \sqrt{37}$, and $\sqrt{37}$ is real and irrational.

Similarly, we see that the roots of $2x^2 + 5x - 8 = 4x - 11$ are imaginary, because for this equation $b^2 - 4ac = -23$.

EXERCISE CXXI

1. By means of the formula of § 178, solve Exs. 6–17, p. 196.

Without first solving the following equations, tell whether their roots are real, imaginary, rational, equal, etc., and explain:

2. $x^2 - 5x + 6 = 0$.

3. $x^2 - 6x + 9 = 0$.

4. $3t^2 - 11t - 17 = 0$.

5. $3t^2 + 11t + 17 = 0$.

6. $\frac{1}{7}(3x^2 + 2) - \frac{1}{3} = \frac{1}{6}(x - 5)$.

7. $25u^2 - 20u + 7 = 0$.

8. In each of Exs. 4–11, p. 197, determine the character of the roots without solving the equation.

9. For what value of k will the roots of $3x^2 - 10x + 2k = 0$ be equal?

SUGGESTION. The roots will be equal if $(-10)^2 - 4 \cdot 3 \cdot 2k = 0$. Why?

For what values of m will Exs. 10–15 have equal roots?

10. $mx^2 - 6x + 3 = 0$. **13.** $my^2 - 5my + 11 = m$.

11. $\frac{x^2}{2} + 3mx + 7 = 0$. **14.** $y^2(1 - m) + 7y = 9 - 3my$.

12. $3x^2 - 4mx + 2 = 0$. **15.** $-4y^2 - 3y - 3 = m(y + 2y^2 + 4)$.

16. Translate into verbal language the conditions for the character of the roots of $ax^2 + bx + c = 0$.

17. Show that if one root of a quadratic equation is imaginary, then both are imaginary, and each is the conjugate of the other.

18. For what values of k are the roots of $36x^2 - 24kx + 15k = -4$ imaginary?

SOLUTION. Here the discriminant $b^2 - 4ac = (-24k)^2 - 4 \cdot 36(15k + 4) = 144(4k^2 - 15k - 4) = 144(4k + 1)(k - 4)$ [cf. § 64]. Hence the roots are imaginary for those values of k which make $(4k + 1)(k - 4)$ negative, and for those values only. Now $(4k + 1)(k - 4)$ is negative only when one of its factors is positive and the other negative; hence the roots of the given equation are imaginary when k lies between $-\frac{1}{4}$ and 4. (Why?)

19. For what values of k are the roots of $36t^2 - 24kt + 15k = -4$ real? How do the roots compare if $k = -\frac{1}{4}$? $k = 4$?

20. Without actually solving the equation, find the values of m for which the roots of $4m^2x^2 + 12m^2x + 10 = m$ are equal; also those values of m for which the roots are real; also, those for which the roots are imaginary.

180. Relation between roots and coefficients. If we let r and r' represent the roots of $ax^2 + bx + c = 0$, *i.e.*, if

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r' = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

then $$r + r' = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}; \quad (1)$$

and $$r \cdot r' = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{c}{a}. \quad (2)$$

Let the pupil work out (1) and (2) in detail, and translate each of these equations into verbal language.

EXERCISE CXXII

In the following equations, name the sum of the roots, also their product; then check your answers by solving the equations.

1. $x^2+5x-2=0$.
2. $x^2-10x=-16$.
3. $4s^2-6s=3$.
4. $x^2-30x+25=0$.
5. $x^2+px=-q$.
6. $ax^2+2bx+c=0$.

7. In each of Exs. 4–11, p. 197, write down, without solving the equation, the sum and the product of its roots; explain your work in each case.

8. If one root of $x^2+5x-24=0$ is known to be 3, how may the other root be found from the absolute term? from the coefficient of the first power of x? Do the results agree?

9. If one root of *any* given quadratic equation whatever is known, how may the other root be most easily found?

10. What is the sum of the roots of $3m^2x^2+(8m-1)x+5=0$? For what value of m is this sum 3?

11. If one root of $2x^2-3(2k+1)x+9k=0$ is the reciprocal of the other, find the value of k.

HINT. Equate one root to the reciprocal of the other and solve for k.

12. For what value of k will one root of the equation in Ex. 11 be zero? With this value of k, what will be the value of the other root?

13. Answer the questions of Exs. 11 and 12 for the equation $2(k+1)^2x^2-3(2k+1)(k+1)x+9k=0$.

14. Show that if one root of $ax^2+bx+c=0$ (whatever the values of a, b, and c) is double the other, then $2b^2=9ac$.

181. Values of simple expressions containing the roots. If r and r' are the roots of a given quadratic equation, § 180 enables us to find the *value* of such expressions as $\frac{1}{r}+\frac{1}{r'}$, $r^2+r'^2$, etc., without first solving the equation.

E.g., the value of $\frac{1}{r}+\frac{1}{r'}$, for the equation $x^2-5x+3=0$, may be found thus:

$$\frac{1}{r}+\frac{1}{r'}=\frac{r'+r}{rr'};$$

but, for this equation, $r'+r=5$ and $rr'=3$,

therefore $$\frac{1}{r}+\frac{1}{r'}=\frac{5}{3}.$$

Similarly, since $r^2+r'^2=(r+r')^2-2rr'$,
therefore, for the above equation,

$$r^2+r'^2=25-6=19.$$

182. Formation of equations whose roots are given numbers. (i) *Sum and product method.* If r and r' are the roots of the equation $ax^2+bx+c=0$, *i.e.*, of $x^2+\frac{b}{a}x+\frac{c}{a}=0$, then (§ 180) this equation may be written in the form:

$$x^2-(r+r')x+rr'=0.$$

And from this we learn how to write down a quadratic equation whose roots are any two given numbers.

E.g., if the roots are to be 2 and -5, we have

$$-(r+r')=3, \text{ and } rr'=-10,$$

whence the equation is

$$x^2+3x-10=0.$$

(ii) *The factor method.* An equation whose roots are any given numbers may be written down as follows (cf. § 72):

The roots of $(x-r)(x-r')=0$
are evidently r and r'; hence the equation whose roots are 2 and -5 is $(x-2)(x+5)=0$,
i.e., as before, $x^2+3x-10=0$.

EXERCISE CXXIII

1. If r and r' denote the roots of $x^2-7x+12=0$, find, without solving the equation, the value of

$$\frac{1}{r+r'},\ \frac{1}{rr'},\ \frac{1}{r}+\frac{1}{r'},\ r^2+r'^2.$$

Find for each of the following equations the value of $\frac{1}{r}+\frac{1}{r'}$, $r^2+r'^2$, $r-r'$, and $\frac{1}{r}-\frac{1}{r'}$:

2. $t^2-12\,t=-36.$

3. $6\,m^2-m=2.$

4. $3\,y^2-16\,y+5=0.$

5. $x^2+2\,x=4.$

6. $x^2+px+q=0.$

7. $ax^2+bx+c=0.$

8. Solve each of the above equations and thus verify the results in Exs. 2–7.

By each of the methods given in § 182, form the equations whose roots are:

9. $5,\ -3.$

10. $-4,\ 4.$

11. $-a,\ -b.$

12. $\frac{1}{5},\ \frac{2}{3}.$

13. $\frac{3}{4},\ -\frac{5}{6}.$

14. $r,\ -r'.$

15. $\frac{1}{r},\ \frac{1}{r'}.$

16. $r^2,\ r'^2.$

17. $\sqrt{6}\pm 4.$

18. $5\pm 2\sqrt{3}.$

19. $2\pm 5\,i.$

20. $\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}.$

21. $\frac{1\pm\sqrt{5}}{2}.$

22. $5,\ -\frac{19}{3}.$

23. $\sqrt{c}\pm\sqrt{d}.$

24. $r\pm\frac{1}{r'}.$

25. The roots of $x^2-5\,x+2=0$ being r and r', form a new equation whose roots are $\frac{1}{r}$ and $\frac{1}{r'}$. (Cf. Exs. 1 and 15 above.)

26. If r and r' are the roots of $3\,z^2-11\,z-20=0$, form an equation whose roots are $\frac{1}{r}$ and $\frac{1}{r'}$; also one whose roots are r^2 and r'^2; also one whose roots are $r+\frac{1}{r'}$ and $r'+\frac{1}{r}$.

Write the equations whose roots are:

27. $-1,\ 2,\ -5.$

28. $-a,\ -b,\ -c.$

29. $1\pm\sqrt{5},\ 5.$

30. $\pm\sqrt{c},\ c+d.$

183. Factors of quadratic expressions. As in § 182, if r and r' are the roots of the equation $ax^2+bx+c=0$, then

$$x^2+\frac{b}{a}\,x+\frac{c}{a}=x^2-(r+r')\,x+rr'=(x-r)(x-r'),$$

i.e., multiplying by a,

$$ax^2 + bx + c = a(x - r)(x - r');$$

hence, if the roots of the *equation* $ax^2 + bx + c = 0$ are r and r', then the factors of the *expression* $ax^2 + bx + c$ are a, $x - r$, and $x - r'$.

184. A quadratic equation has two roots, and only two. By actually solving the equation $ax^2 + bx + c = 0$ (§ 126) we find that it has two roots, say r and r'. That it can have no other root, as r'', is evident if we write the equation in the form

$$a(x - r)(x - r') = 0 \qquad [§ 183$$

and observe that $a(r'' - r)(r'' - r')$ cannot be zero if r'' differs from both r and r'.

EXERCISE CXXIV

1. Since 2 and 7 are the roots of $x^2 - 9x + 14 = 0$, what are the factors of $x^2 - 9x + 14$?

2. By first finding the roots of the *equation* $15x^2 - 4x - 3 = 0$, find *all* the factors of the *expression* $15x^2 - 4x - 3$. Check your answer by finding the product of these factors.

3. Write a carefully worded rule for factoring quadratic expressions by the method of § 183.

Find (in accordance with the rule just made) *all* the factors of the following expressions, and check your results:

4. $5x^2 - 12x - 9.$

5. $8z^2 - 45z - 18.$

6. $x^2 + 9.$

7. $s^2 + s + 1.$

8. $4m^2 - 24m - 13.$

9. $8x^2 - 2x - 3.$

10. $(x + 1)(2 - x) + 9 - x.$

11. $(2y - 1)^2 - 5(y + 1) + 8.$

12. $x^2 + px + q.$

13. $ax^2 + bx + c.$

14. $7x^2 - \dfrac{13x}{5} - \dfrac{4}{5}.$

15. $\dfrac{3m^2}{4} - m - 5.$

16. $m^2 + 6m + 13.$

17. Are the expressions in Exs. 4–16 equal to 0? What justification have we, then, for writing them so?

18. How many roots has a quadratic equation? Verify your answer for the equations $28x^2+29x+6=0$ and $m^2-10m=-25$.

19. Show that the cubic equation $27y^3-1=0$ has three roots and only three (cf. Ex. 17, p. 263).

REVIEW EXERCISE — CHAPTERS XI–XVII

1. Find the square root of $x^6+20x^2+16-4x^5+16x$; also of $x^4-2x^2y^{-1}-4x^{\frac{1}{2}}+y^{-2}+4x^{\frac{5}{2}}y+4xy^2$.

2. Find (correct to three decimal places) the value of $\sqrt{11.7}$; $\sqrt{\frac{7}{18}}$; $\sqrt{23561}$.

3. Expand by the binomial theorem: $(x^3-y^2)^{11}$; $\left(\frac{m}{2}+3n\right)^5$; $(1+c^{-2})^7$; $(1-3m^{\frac{1}{2}})^5$; $\left(1-\frac{1-x^{\frac{1}{2}}}{1+x^{\frac{1}{3}}}\right)^3$.

4. Use the binomial theorem to find (correct to five decimal places) the value of $(10.001)^7$, *i.e.*, of $(10+.001)^7$.

Simplify:

5. $\sqrt[3]{\frac{1}{4}}+\sqrt[3]{\frac{1}{32}}+\sqrt[3]{\frac{2}{3}}$.

6. $a\sqrt{\dfrac{b^2z+b^2}{z-1}}-b\sqrt{\dfrac{a^2z-a^2}{z+1}}$.

7. $\sqrt[3]{abc^2}(a^{\frac{4}{3}}b^{-1}c^{\frac{1}{3}})(a^{-\frac{2}{3}}b^{\frac{2}{3}}c^{-1})$.

8. $\sqrt[mn]{\dfrac{c^m}{c^n}}\cdot\sqrt[np]{\dfrac{c^n}{c^p}}\cdot\sqrt[pm]{\dfrac{c^p}{c^m}}$.

9. $(a^3-b^2)\div(\sqrt[4]{a^3}-\sqrt{b})$.

10. $\sqrt{\dfrac{c+d}{c-d}}\sqrt{\dfrac{c+d}{c-d}}\cdot\sqrt[3]{\dfrac{c-d}{c+d}}\sqrt{\dfrac{c-d}{c+d}}$.

11. $(2\sqrt{-3}-3\sqrt{-2})(6\sqrt{-2}+4\sqrt{-3})$.

12. Show that $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^3+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^3=2$.

13. By rationalizing the divisor perform the following indicated divisions:

$$\frac{1+r+\sqrt{1-r^2}}{1+r-\sqrt{1-r^2}};\quad \frac{1}{\sqrt{6}-\sqrt{3}-1};\quad \frac{\sqrt{x}-\sqrt{-3y}}{\sqrt{-x}-\sqrt{-y}};\quad \frac{3}{3^{\frac{1}{2}}+5^{-\frac{1}{3}}}.$$

14. Express $\sqrt[4]{89-28\sqrt{10}}$ as the difference of two surds.

15. Find the sum of $\sqrt{-25}$, $-\frac{1}{2}\sqrt{-9}$, $5-10\sqrt{-\frac{1}{4}}$, and $-2-7i$, graphically; also, by the same method, subtract the fourth of these numbers from the third.

Solve the following equations and check as the teacher directs:

16. $\dfrac{x^2}{2}-\dfrac{x}{3}+7\tfrac{3}{8}=8.$

17. $\sqrt{x}-\sqrt{1+x}=\dfrac{1}{\sqrt{x}}.$

18. $\dfrac{s}{4}-\dfrac{21-s}{4-s}=1.$

19. $\dfrac{3x+\sqrt{4x-x^2}}{3x-\sqrt{4x-x^2}}=2.$

20. $m+5+\sqrt{m+5}=6.$

21. $\sqrt{a+z}+\sqrt{a-z}=\sqrt{b}.$

22. $x^4+4x^2-117=0.$

23. $19x^4+216x^7=x.$

24. $144r^2-1+6\sqrt{9r^2-r}=16r.$

25. $(7-4\sqrt{3})x^2+(2-\sqrt{3})x-2=0.$

Solve the following systems:

26. $\begin{cases} x^2+y^2=85, \\ xy=42. \end{cases}$

27. $\begin{cases} x^3-y^3=215, \\ x^2+xy+y^2=43. \end{cases}$

28. $\begin{cases} 3r^2-2rs=15, \\ 2r+3s=12. \end{cases}$

29. $\begin{cases} xy=4-y^2, \\ 2x^2-y^2=17. \end{cases}$

30. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=7, \\ \dfrac{1}{x^2}+\dfrac{1}{y^2}=29. \end{cases}$

31. $\begin{cases} vw+\sqrt{v+w}=11, \\ 2vw-\sqrt{v+w}=13. \end{cases}$

32. Show that the difference of the roots of the equation $x^2+px+q=0$ is the same as that of $x^2+3px+2p^2+q=0$.

33. For what values of m will $x^2-2(1+3m)x+7(3+2m)=0$ have real roots? equal roots? imaginary roots?

34. Form the equation whose roots are a, $-a$, and b; also the equation whose roots are the negative reciprocals of the roots of $ax^2-bx+c=0$.

35. If the roots of $x^2-px+q=0$ are two consecutive integers, show that $p^2-4q-1=0$.

36. A rectangular plot of ground contains 42 acres; find its sides if its diagonal measures 1243 yards.

37. In a regiment drawn up in the form of a solid square, the number of men in the outside five rows is $\frac{9}{25}$ of the entire regiment. Find the size of the regiment.

38. In a quarter of a mile drive, the fore wheel of a carriage makes 22 revolutions more than the hind wheel; if the circumference of each wheel were 2 ft. less than it now is, the fore wheel would make 33 revolutions more than the hind wheel. Find the circumference of each.

39. A crew can row a certain course upstream in $8\frac{4}{7}$ minutes, and were there no current, they could row it in 7 minutes less than the time it now takes them to drift downstream. How long would it take them to row the course downstream?

40. Two men, A and B, have a money box containing $210, from which each draws a certain fixed sum daily, the two sums being different. Find the sum drawn daily by each, knowing that A *alone* would empty the box 5 weeks earlier than B *alone*, while the two together empty it in 6 weeks.

CHAPTER XVIII

INEQUALITIES

185. Definitions. The symbols $>$ and $<$ stand for "is greater than," and "is less than," respectively; thus, the expression $a<b$ is read "a is less than b." One real number, a, is said to be **greater** than another, b, if $a-b$ is positive; if $a-b$ is negative then a is **less** than b.

E.g., $5<8$, since $5-8=-3$; and $-4>-9$, since $-4-(-9)=+5$.

A statement that one of two numbers is greater or less than the other is called an **inequality**; thus, $5x-3>2y$ is an inequality, of which $5x-3$ is the first member, and $2y$ the second.

Two inequalities are said to be of the **same species** (or to *subsist in the same sense*) if the first member is the greater in each, or if the first member is the less in each; otherwise they are of **opposite species**.

Thus, the inequalities $a>b$ and $c+d>e$ are of the same species, while $x^2+y^2>z^2$ and $m^2<n^2+mn$ are of opposite species.

186. General principles in inequalities. Before memorizing the following principles (1–7), the pupil should illustrate each by one or more numerical examples; he should also try to invent a proof of his own for each principle before reading the printed proof.

PRINCIPLE 1. *If the same number is added to, or subtracted from, each member of an inequality, the result is an inequality of the same species.*

For, if $a < b$, *i.e.*, if $a - b$ is negative, then $a + c - (b + c)$, which equals $a - b$, is negative, and therefore

$$a + c < b + c;$$

and similarly $$a - c < b - c.$$

So, too, if $a > b$, then $a + c > b + c$, and $a - c > b - c$.

PRINCIPLE 2. *If each member of an inequality is multiplied or divided by the same positive number, the result is an inequality of the same species.*

For, if $a > b$, and n is any positive number, then $an - bn$, *i.e.*, $(a - b)n$, is positive (why?), and therefore $an > bn$.

Similarly if we divide by n; and so, too, if $a < b$.

PRINCIPLE 3. *If each member of an inequality is multiplied or divided by the same negative number, the result is an inequality of opposite species.*

HINT. If $a > b$, and n is negative, then $an - bn$ is negative (why?) and therefore $an < bn$.

PRINCIPLE 4. *If several inequalities of the same species are added, member to member, the result is an inequality of the same species.*

HINT. Let $a < b$, $c < d$, $e < f$, $\cdots$, then $(a - b) + (c - d) + (e - f) + \cdots$, *i.e.*, $(a + c + e + \cdots) - (b + d + f + \cdots)$, is negative (why?), and therefore $a + c + e + \cdots < b + d + f + \cdots$.

PRINCIPLE 5. *If an inequality is subtracted from an equation, or from an inequality of opposite species, member from member, the result is an inequality whose species is opposite to that of the subtrahend.*

HINT. Let $a < b$, and $c = d$ or $c > d$, then in either case, $b - a + c - d$, which equals $c - a - (d - b)$, is positive (why?), and therefore $c - a > d - b$.

PRINCIPLE 6. *If the first of three numbers is greater than the second, and the second is greater than the third, then the first is greater than the third; and conversely.*

HINT. Let $a > b$ and $b > c$, then $(a - b) + (b - c)$, *i.e.*, $a - c$, is positive (why?), and therefore $a > c$.

PRINCIPLE 7. *If two or more inequalities of the same species, whose members are positive, are multiplied together, member by member, the result is an inequality of the same species.*

HINT. Let $a > b$ and $c > d$, then, by Prin. 2, $ac > bc$ and $bc > bd$, whence, by Prin. 6, $ac > bd$; and similarly for three or more such inequalities.

187. Conditional and unconditional inequalities. An identical or **unconditional inequality** is one which is true for all values of its letters. Thus, $a + 4 > a$ and $(x - y)^2 + 1 > 0$ are unconditional inequalities.

A **conditional inequality** is one which is true only on condition that certain restricted values are assigned to its letters. Thus, $x + 4 < 3x - 2$ only on condition that $x > 3$.

A conditional inequality is solved by means of the principles of § 186, and in much the same way that an equation is solved by means of the ordinary axioms.

Ex. 1. If $3x - \frac{25}{3} > \frac{11}{3} - x$, find the possible values of x.

SOLUTION. On multiplying each member of this inequality by 3, it becomes

$$9x - 25 > 11 - 3x, \qquad [\S\ 186, 2$$

whence $$9x + 3x > 11 + 25, \qquad [\S\ 186, 1$$

i.e., $$12x > 36,$$

whence $$x > 3; \qquad [\S\ 186, 2$$

i.e., if the given inequality is true, x must be greater than 3.

By means of the principles established in § 186 the student may show that each step in the above reasoning is reversible, and hence that the converse is also true; viz., that if $x > 3$, then $3x - \frac{25}{3} > \frac{11}{3} - x$.

Ex. 2. Given the two relations $\begin{cases} 2x + 3y > 5, & (1) \\ x + 4y = 6; & (2) \end{cases}$

to find those values of x and y that will satisfy them *both*.

SOLUTION. On multiplying each member of (1) by 4, and each member of (2) by 3, we obtain

$$8x + 12y > 20,$$

and $$3x + 12y = 18;$$

whence, subtracting, $$5x > 2, \qquad [\S\ 186, 1$$

and therefore $$x > \tfrac{2}{5}. \qquad [\S\ 186, 2$$

Now substitute for x in (2) above, any number greater than $\frac{2}{5}$, and find the corresponding value of y (this value of y will always be less than $\frac{7}{5}$; why?); these values of x and y, taken together, will satisfy both (1) and (2).

EXERCISE CXXV

3. Using the definitions of "greater" and "less" in § 185, show that $5 > 2$; that $-23 < -12$; and that $2 > -9$.

4. If $x + y > z - w$, show that $x + w > z - y$.

HINT. Apply Principle 1 twice.

5. May terms be transposed from one member of an inequality to the other? If so, how and why (cf. Ex. 4)?

6. If $\frac{m-n}{2} < \frac{m+2n}{3}$, show that $3(m-n) < 2(m+2n)$. How may an inequality be cleared of fractions? Why?

7. Show, from Principles 2 and 3, how to remove a common factor from both members of an inequality.

8. What happens if the signs in each member of an inequality are reversed? Why?

HINT. In the proof of Principle 3, put -1 for n.

9. If $a > b$ and $c = d$, show that $c - a < d - b$.

10. Illustrate by numerical examples that

(1) if $a > b$ and $c < d$, then the sum of these inequalities may be either $a + c = b + d$, or $a + c > b + d$, or $a + c < b + d$;

(2) if $a > b$ and $c > d$, then the difference of these inequalities may be either $a - c = b - d$, $a - c > b - d$, or $a - c < b - d$.

11. Translate (1) and (2) of Ex. 10 into verbal language.

12. If $a < b$ and $c < d$, and if d alone is positive, show that $ac > bd$. Is this inconsistent with Principle 7?

13. If a, b, c, and d are positive numbers, and if $a > b$ while $c < d$, which is the greater, ac or bd? Why? Illustrate your answer numerically.

14. What operations with or upon inequalities lead to results, one of which is certainly greater or less than the other?

15. Name and illustrate some operations with inequalities which lead to results whose relations are uncertain.

16. Show that $a^2 + b^2 > 2ab$ except when $a = b$.

HINT. $(a - b)^2$ is positive whether $a > b$ or $a < b$.

17. Distinguish between a conditional and an unconditional inequality. To which of these classes does $a^2 + b^2 + 1 > 2ab$ belong? Why?

18. To which class of inequalities does $6x - 5 > 3x$ belong? Why? Solve this inequality.

19. If $3x < 5x - 9$, show that x is greater than $4\frac{1}{2}$ (cf. Ex. 1).

20. If $x^2 + 24 < 11x$, show that the range of values of x is between 3 and 8, *i.e.*, that x must be greater than 3 and less than 8.

HINT. In order that $(x - 3)(8 - x)$, *i.e.*, $11x - x^2 - 24$, may be positive, both factors must be positive or both negative.

Find the range of values of x in each of Exs. 21–26:

21. $x^2 < 9$.

22. $x^2 + 24 > 11x$.

23. $30 > x + \dfrac{3x}{2} > 25$.

24. $28 > 3x + x^2$.

25. $\begin{cases} 4x - 11 > \dfrac{x}{3}, \\ 10 - x > 5. \end{cases}$

26. $\begin{cases} 3 - 4x < 7, \\ x + 2 < 4. \end{cases}$

27. Show that no positive number plus its reciprocal is less than 2; *i.e.*, n being any positive number, that $n + \dfrac{1}{n} \nless 2$.*

28. Show that $4x^2 + 9 \nless 12x$.

29. Show that $2b(6a - 5b) \ngtr (2a + b)(2a - b)$.

If a, b, and c are positive and unequal, show that

30. $a^4 + b^4 > a^2b^2$.

31. $\dfrac{a + 2b}{a + 3b} < \dfrac{a + 3b}{a + 4b}$.

32. $a^3 + b^3 > a^2b + ab^2$.

33. $a^2 + b^2 + c^2 > ab + bc + ca$.

34. $a^3 + b^3 + c^3 > 3abc$ (cf. Ex. 30, p. 51).

35. If $a^2 + b^2 = 1$, and $c^2 + d^2 = 1$, prove that $ab + cd \ngtr 1$.

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* Cf. Ex. 16. The symbol $\nless$ stands for "is not less than," and $\ngtr$ stands for "is not greater than."

36. If both m and n are positive, which is the greater, $\frac{m+n}{2}$ or $\frac{2\,mn}{m+n}$?

Solve the following systems:

37. $\begin{cases} 2\,x - 3\,y < 2, \\ 2\,x + 5\,y = 6. \end{cases}$

38. $\begin{cases} 3\,x + 2\,y = 42, \\ 3\,x - \frac{y}{7} > 16. \end{cases}$

39. $\begin{cases} y - x > 9, \\ \frac{7\,x}{20} + \frac{y}{15} = 9. \end{cases}$

40. $\begin{cases} x + y = 10, \\ 4\,x < 3\,y. \end{cases}$

41. Find the smallest integer fulfilling the condition that $\frac{1}{3}$ of it decreased by 7 is greater than $\frac{1}{4}$ of it increased by 6.

42. The sum of three times A's money and 4 times B's is \$1 more than 6 times A's; and if A gives \$5 to B, then B will have more than 6 times as much as A will have left. Find the range of values of A's money and B's.

CHAPTER XIX

RATIO, PROPORTION, AND VARIATION

I. RATIO

188. Definitions. The **ratio (direct ratio)** of two numbers is the quotient obtained by dividing the first of these numbers by the second. The numbers themselves are usually called the **terms** of the ratio, the first being the **antecedent**, and the second the **consequent**.

E.g., the ratio of 15 to 5 is $15 \div 5$, *i.e.*, 3; the antecedent is 15, and the consequent is 5.

The ratio of a to b may be written as $a : b$, $a \div b$, or $\frac{a}{b}$; it is read "the ratio of a to b," or "a divided by b."

The **inverse ratio** of a to b is $b \div a$, *i.e.*, it is the reciprocal of the direct ratio of these numbers.

Two numbers are said to be **commensurable** or **incommensurable** with each other according as their ratio is rational or irrational (cf. § 146), *i.e.*, according as they have or have not a common measure.

E.g., 1.5 and $\frac{2}{3}$ are commensurable with each other; so also are $3\sqrt{2}$ and $5\sqrt{2}$; but $3\sqrt{2}$ and 5 are incommensurable.

189. Ratio of like quantities. The concrete *quantities* with which algebra is concerned are expressed by means of *numbers*, and the ratio of two *like* * quantities is therefore the ratio of the numbers which represent these quantities.

E.g., the ratio $\$6 : \$9$ is the same as $6 : 9$, *i.e.*, as $2 : 3$.

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* Unlike quantities can, of course, have no ratio to each other.

190. Properties of ratios. Since ratios are quotients, *i.e.*, fractions, therefore they have all the properties of fractions.

Thus, a ratio is not changed if both antecedent and consequent are multiplied or divided by any given number.

Again, if a, b, and k are positive, and $a<b$, then since

$$\frac{a}{b}-\frac{a+k}{b+k},\ i.e.,\ \frac{(a-b)k}{b(b+k)}, \qquad [\S\ 88$$

is negative ($a-b$ being negative), therefore (§ 185), *the ratio* $a+k:b+k$ *is greater than the ratio* $a:b$. [Translate this important fact into words: (1) calling $a \div b$ a proper fraction; and (2) calling $a \div b$ a ratio less than unity.]

EXERCISE CXXVI

What is the ratio of:

1. 6 to 8? **3.** -10 to 24? **5.** $-\frac{3}{8}$ to $\frac{15}{16}$?

2. 50 to 15? **4.** 6.3 to .7? **6.** $21x^2$ to $9x$?

7. Which term of a ratio corresponds to the divisor? What is the other term called? Illustrate from Exs. 1–6.

8. Form the inverse of each of the ratios in Exs. 1–6.

9. The ratio of x to 5 equals 2; find x, and check your work.

10. If the ratio of two numbers is $\frac{2}{3}$ and the consequent is 6, find the antecedent.

In each of Exs. 11–14, find (and check) the value of x:

11. $x^2 : 2 = \frac{2}{9}$. **13.** $64 : x = x$.

12. $x+5 : 2x = -7$. **14.** $25 : x^2 = 9$.

15. What is the ratio of x to y when $7(x-y)=3(x+y)$? when $x^2+6y^2=5xy$?

16. A yard measure is divided into two parts whose lengths are in the ratio 7 : 11; how many inches in each part?

17. A and B divide \$100 between them so that A receives \$13 out of every \$20. What is the ratio of A's share to B's share? How many dollars does each receive?

18. What number must be added to each term of $\frac{15}{27}$ in order that the resulting ratio shall be $2:3$? Does this addition increase or diminish the given ratio?

19. Which is the greater ratio, $5+2:17+2$ or $5:17$? $21+8:11+8$ or $21:11$? Does the addition of the same positive number to both terms of a ratio *always* increase the latter's value (cf. § 190)? Explain.

20. If a, b, and k represent positive numbers, translate (1) and (2) below into words; then give three numerical illustrations of each:

$$(1)\quad \frac{a-k}{b-k}<\frac{a}{b}\text{ when } a<b;$$

$$(2)\quad \frac{a-k}{b-k}>\frac{a}{b}\text{ when } a>b.$$

21. By a method similar to that used in § 190, show the correctness of (1) and (2), Ex. 20.

22. If x, y, and z are positive numbers, which is the greater ratio (and why?), $\frac{2x+5y}{2x+7y}$ or $\frac{x+2y}{x+3y}$? $\frac{x-y+z}{x+y-z}$ or $\frac{x+y+z}{x-y-z}$?

23. Show that the following ratios are all equal: $12 : $9; 8 bu. oats : 6 bu. oats; 4 T. of coal : 3 T. of coal; 10 in. : $7\frac{1}{2}$ in.; $4:3$; $\frac{1}{5}:\frac{3}{20}$.

24. Find the value of each of the following ratios: $4\sqrt{2}:\sqrt{2}$; $4\sqrt{2}:2$; $7\sqrt{3}$ in. : $14\sqrt{2}$ in.; $5.80 : 29 cents.

25. Find two integers whose ratio equals $15\frac{3}{8}:9\frac{5}{9}$. Can the ratio of any two numbers whatever be expressed as the ratio of two integers (cf. Ex. 24, also § 188)?

26. Which of the pairs of numbers (or quantities) in Ex. 24 are commensurable? Which are incommensurable? Why?

II. PROPORTION

191. Definitions. If a, b, c, and d are any four numbers such that $a:b=c:d$, then these numbers are said to be proportional, or to form a proportion; *i.e.*, a **proportion** is a statement that two ratios are equal.

The proportion $a:b=c:d$ (sometimes written $a:b::c:d$) is read: "the ratio of a to b equals the ratio of c to d," and also "a is to b as c is to d." In this proportion a and d are called the **extremes**, while b and c are called the **means**.

If $a:b=c:d$, then d is said to be the **fourth proportional** to a, b, and c; while if $a:b=b:c$, then c is called the **third proportional** to a and b, and b is called the **mean proportional** between a and c.

A succession of equal ratios, in which the consequent of each is also the antecedent of the next, is called a **continued proportion**; thus $a:b=b:c=c:d=d:e=\cdots$, is a continued proportion.

EXERCISE CXXVII

1. Is it true that $8:12=10:15$? Why? How is this proportion read? What does it mean?

2. In Ex. 1 name the means and the extremes of the proportion, also the fourth proportional to 8, 12, and 10.

3. Is it true that $8:10=12:15$? How does this proportion compare with that in Ex. 1? Does a proportion remain true after its means have been interchanged? Try several numerical examples and compare Principle 4, p. 298.

4. By arranging the numbers 3, 4, 6, and 8 in different ways, make three different proportions.

5. Is 6 a mean proportional between 4 and 9? between 18 and 2? Is the same thing true of -6? Name the third proportional in each case.

6. Show that $2:6=6:18=18:54=54:162$ is a continued proportion in which each ratio equals $\frac{1}{3}$. Write a continued proportion of five ratios each of which equals $\frac{3}{4}$.

192. Important principles of proportion. Since a proportion is merely an *equation* whose members are *fractions*, therefore the principles of proportion may be derived from those governing equations and fractions.

NOTE. Before memorizing the following principles (1–8) the pupil should illustrate each by one or more numerical examples; he should also try to *invent* a proof of his own for each principle, before reading the printed proof.

PRINCIPLE 1. *If four numbers are in proportion, then the product of the extremes equals the product of the means.*

For, let a, b, c, and d be any four numbers which are in proportion, then $a : b = c : d$;

i.e., $\frac{a}{b} = \frac{c}{d}$,

whence $ad = bc$, [clearing of fractions

which was to be proved.

PRINCIPLE 2. *If the product of two numbers equals the product of two others, then either pair may be made the extremes of a proportion in which the other pair are the means.*

For, if $ad = bc$,

then $\frac{a}{b} = \frac{c}{d}$, [dividing by bd

i.e., $a : b = c : d$.

In the same way it may be shown that, if $ad = bc$, then

$$b : a = d : c, \quad c : a = d : b, \text{ etc.}$$

REMARK. From the proof just given it follows that *the correctness of a proportion is established when it is shown that the product of the means equals the product of the extremes*; this test is very useful.

PRINCIPLE 3. *If four numbers are in proportion, then they are in proportion by* **inversion**; i.e., *the second is to the first as the fourth is to the third.*

For, if $a : b = c : d$, then $ad = bc$ (why?); hence $b : a = d : c$ (cf. Principle 2, Remark).

SUGGESTION. Let the pupil state Principles 4–7 below entirely in verbal language, and prove each in detail (cf. statement and proof of Principle 2).

PRINCIPLE 4. *If four numbers are in proportion, then they are in proportion by* **alternation**; *i.e.*, if $a : b = c : d$, then $a : c = b : d$.

PRINCIPLE 5. *If four numbers are in proportion, then they are in proportion by* **composition**; *i.e.*, if $a : b = c : d$, then $(a + b) : a = (c + d) : c$; [also, $(a + b) : b = (c + d) : d$].

PRINCIPLE 6. *If four numbers are in proportion, then they are in proportion by* **division** *or* **separation**; *i.e.*, if $a : b = c : d$, then $(a - b) : a = (c - d) : c$; [also, $(a - b) : b = (c - d) : d$].

PRINCIPLE 7. *If four numbers are in proportion, then they are in proportion by* **composition and division**; *i.e.*, if $a : b = c : d$, then $(a + b) : (a - b) = (c + d) : (c - d)$.

PRINCIPLE 8. *In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its own consequent.*

Thus, if $a : b = c : d = e : f = g : h = \cdots = x : y$,

then $(a + c + e + g + \cdots + x) : (b + d + f + h + \cdots + y) = e : f$.

To prove this theorem, let each of the given equal ratios be represented by a single letter, say r;

then $$\frac{a}{b} = r,\ \frac{c}{d} = r,\ \frac{e}{f} = r,\ \frac{g}{h} = r,\ \cdots, \text{ and } \frac{x}{y} = r,$$

hence $a = br$, $c = dr$, $e = fr$, $g = hr$, $\cdots$, and $x = yr$,

and, adding these equations, member to member,

$$a + c + e + g + \cdots + x = (b + d + f + h + \cdots + y)r,$$

and therefore $$\frac{a + c + e + g + \cdots + x}{b + d + f + h + \cdots + y} = r = \frac{e}{f},$$

which proves the principle.

NOTE. As in the proof just given, so it will often be found advantageous to represent a ratio by a single letter.

EXERCISE CXXVIII

Using Principle 1, find x in each of Exs. 1–4:

1. $14 : 3 = 56 : x$.
2. $x : -5 = 20 : -2$.
3. $-16 : x = 18 : 7$.
4. $\frac{7}{8} : x = x : \frac{7}{32}$.

Find a mean proportional between each of the following pairs of numbers (cf. Ex. 4):

5. $3, 27$.
6. $-2, -5$.
7. $25, -4$.
8. $.25, .09$.
9. am^{-3}, a^3m.
10. $a + b, a - b$.

11. How many answers has each of Exs. 5–10? Why? Show that the mean proportional between any two numbers equals the square root of their product.

12. Find the third proportional to 1 and 4; to -25 and -40; also the fourth proportional to $m - n$, $m^2 - n^2$, and $m + n$.

13. Using Principles 2–7, make seven different proportions from the equation $cd = mn$.

14. Add 1 to each member of the equation $a : b = c : d$; write the result as a proportion and thus prove Principle 5.

15. Prove that like powers (also like roots) of proportional numbers are proportional, *i.e.*, prove that if $a : b = c : d$, then $a^n : b^n = c^n : d^n$.

16. If $a : b = c : d$ and $e : f = g : h$, show that $ae : bf = cg : dh$; also translate this principle into verbal language.

17. If $a : b = c : d$ and $e : f = g : h$, show that $\frac{a}{e} : \frac{b}{f} = \frac{c}{g} : \frac{d}{h}$.

HINT. Use a single letter to represent a ratio (cf. proof of Principle 8).

If $p : q = r : s$, and m and n are any numbers whatever, show that the proportions in Exs. 18–25 are true.

18. $mp : nq = mr : ns$ (cf. § 190).
19. $5p : r = 5q : s$.
20. $r : s = \frac{1}{q} : \frac{1}{p}$.
21. $s^3 : q^3 = r^3 : p^3$.
22. $p + q : 2p = r + s : 2r$.
23. $pr : qs = r^2 : s^2$.
24. $p^n - 4r^n : q^n - 4s^n = -\frac{r^n}{3} : -\frac{s^n}{3}$.

25. $p+q:r+s=\sqrt{p^2+q^2}:\sqrt{r^2+s^2}$.

26. If $a:b=c:d=e:f=g:h=\cdots$, and $l, m, n, p, \cdots$ are any numbers whatever, show that

$$(ma+lc-ne+pg+\cdots):(mb+ld-nf+ph+\cdots)=a:b.$$

HINT. Compare § 190, also Principle 8.

27. If $(p+q+r+s)(p-q-r+s)=(p-q+r-s)(p+q-r-s)$, show that $p:q=r:s$.

28. If $\dfrac{ax+cy}{by+dz}=\dfrac{ay+cz}{bz+dx}=\dfrac{az+cx}{bx+dy}$, show that each of these ratios equals $\dfrac{a+c}{b+d}$ (cf. Principle 8).

By the principles of proportion solve the equations:

29. $x:15=x-1:12$.

30. $x^2:32=x+2:12$.

31. $(\frac{1}{2}cx+1):3x^2=\frac{1}{2}c^2$.

32. $\begin{cases} x-y=2, \\ \dfrac{x^2+y^2}{(x+y)^2}=\dfrac{5}{9}. \end{cases}$

33. $x:27=y:9=2:x-y$.

34. $\dfrac{x+\sqrt{x-1}}{x-\sqrt{x-1}}=\dfrac{13}{7}$.

HINT. Apply Principle 7.

35. $\dfrac{\sqrt{x+7}+\sqrt{x}}{\sqrt{x+7}-\sqrt{x}}=\dfrac{4+\sqrt{x}}{4-\sqrt{x}}$.

36. What number must be added to each of the numbers 7, 9, 11, and 21 in order that the four sums may be proportional?

37. In the triangle ABC, AK divides BC into two parts, BK and KC, respectively proportional to AB and AC. If $AB=10$ in., $AC=16$ in., and $BC=20$ in., find BK and KC. (Draw a figure to illustrate.)

38. Find two different numbers, m and n, such that

$$m+n:m-n:m^2+n^2=5:3:51.*$$

39. The perimeter of a triangle whose sides are in the ratio 5 : 6 : 8, is 57 meters; find the lengths of the sides.

40. How may \$10 be divided among three boys so that for every dollar the first receives, the second shall receive 15 cents and the third 10 cents?

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* The expression $a:b:c=x:y:z$, means that $a:b=x:y$, $a:c=x:z$, and $b:c=y:z$; and also the equivalent statement $a:x=b:y=c:z$.

41. Two rectangles are equal in area. If their widths are as $2:3$, find the ratio of their lengths.

42. The sides of a certain rectangle are in the ratio $7:3$. Compare the area of the rectangle with that of a square which has the same perimeter.

43. If $a:b$, $c:d$, $e:f$, $g:h$, $\cdots$ are unequal ratios, in which a, b, c, $\cdots$ are positive numbers, and if $a:b$ is the greatest and $e:f$ the least among these ratios, show that

$$(a+c+e+g+\cdots):(b+d+f+h+\cdots)$$

is less than $a:b$, but greater than $e:f$ (cf. proof of Principle 8).

III. VARIATION

193. Variables, constants, and limits. Many questions in mathematics are concerned with numbers whose values are changing; such numbers are usually called **variables**, while numbers whose values do not change are called **constants.**

If the difference between a variable (in the course of its changes) and a constant may become and remain less than any assigned number however small, then this constant is said to be the **limit** of the variable.

Thus, your own age, the height of the mercury column in a thermometer tube, the length of the shadow cast by a given flagstaff, etc., are variables; while the difference between the ages of two given men, the weight of the mercury in a given thermometer, the length of a certain flagstaff, etc., are constants.

Again, the decimal $.3333\cdots$ (*i.e.*, $.3+.03+.003+\cdots$) is a variable whose limit is $\frac{1}{3}$; this decimal grows larger and larger as more and more places are included, and may thus be made to differ from $\frac{1}{3}$ by less than any assigned number however small.

So, too, $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\cdots$ is a variable whose limit is 2 (cf. § 202).

194. Interpretation of the forms $\dfrac{a}{0}$, $\dfrac{a}{\infty}$, and $\dfrac{0}{0}$. Two of these forms were first met with in § 41, and were there interpreted by assuming that the definition of division, given in § 8, remains valid for infinitely large numbers and for zero.

It is better, however, to interpret these forms from the standpoint of variables and limits.

(i) If the values 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, ⋯, are successively assigned to x, what are the corresponding values of $\frac{6}{x}$? of $\frac{a}{x}$, where a is any finite constant whatever? Answer the same questions when x takes the successive values 1, 10, 10^2, 10^3, ⋯.

These examples illustrate two important facts, viz.:

(1) *As the divisor grows smaller and smaller, approaching zero as a limit (the dividend being a finite constant), the quotient increases without limit.*

(2) *As the divisor increases without limit (the dividend being a finite constant), the quotient approaches zero as a limit.*

For the sake of brevity, (1) and (2) are often expressed by the equations

$$\frac{a}{0}=\infty \text{ and } \frac{a}{\infty}=0,$$

respectively; but the interpretation of these equations is as stated in (1) and (2) above.

(ii) In the fraction $\frac{3x^2-3}{x-1}$, as x takes the successive values 1.1, 1.01, 1.001, 1.0001, what limit is approached by the numerator? by the denominator? by the *value* of the fraction? Answer the same questions for $\frac{x-2}{x^2-4}$, as x approaches 2 as a limit.

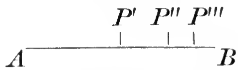
These examples illustrate the fact that as a dividend and its divisor each approach zero as a limit, the quotient may approach any value whatever; this is often expressed by saying that

$$\frac{0}{0} \textit{ is indeterminate.}$$

EXERCISE CXXIX

1. Which of the following quantities are constants and which are variables: (1) the circumference of a growing orange? (2) the length of the shadow cast by a certain church steeple between sunrise and sunset? (3) the length of the steeple itself? (4) the time since the discovery of America? (5) the interest earned by an outstanding note? (6) the principal of the note?

2. A point P moves through half the distance AB (*i.e.*, to P'), then through half the remaining distance (*i.e.*, to P''), then through half the remaining distance (*i.e.*, to P'''), and so on. Show that the distance from A to P is a variable whose limit is AB.



3. In Ex. 2, is the distance from P to B a constant or a variable? What is its limit? Explain both answers.

4. If x takes in succession the values .6, .06, .006, .0006, $\cdots$, what is its limit? Why?

5. What is the limit of the variable sum $.6 + .06 + .006 + \cdots$ (*i.e.*, of the decimal $.6666\cdots$)? Explain.

6. If r is any finite constant, trace the changes in the quotient r/s as s passes through the values, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\cdots$; also as s passes through the values 3, 9, 27, 81, $\cdots$. Is there a limit to the quotient in the first case? in the second? Explain.

7. Translate into verbal language [cf. § 194 (i)]:

$$(1)\ \frac{a}{0} = \infty; \qquad (2)\ \frac{a}{\infty} = 0.$$

8. As x approaches the limit 1, what limit does $\dfrac{x^2-1}{x-1}$ approach in *form*? in *value*? Answer the same questions for the fractions

$$\frac{x-1}{x^2-1},\quad \frac{x^2-x}{x-1},\quad \text{and}\quad \frac{3x^2-2x-1}{x^2-1}.$$

9. By means of your answers to the questions in Ex. 8, illustrate the fact that $\dfrac{0}{0}$ is indeterminate in value.

195. Direct and inverse variation; etc. Of two variables which are so related that, during all their changes, their ratio remains constant, each is said to **vary** (also to **vary directly**) as the other. The symbol employed to denote variation is $\propto$; it stands for the words "varies as," and the expression $a \propto b$ is read "a varies as b."

If $a \propto b$, *i.e.*, if $a : b = k$, a constant, then $a = kb$ (why?); hence a variation statement may be converted into an equation.

E.g., if a tank contains v cu. ft. of water, each cubic foot weighing 62.5 lb., and if the total weight of the water is w lb., then:

(1) When v changes (as it must, for example, while the tank is filling), w changes also.

(2) Since, no matter how the quantity of water changes, $w = 62.5\,v$, or $w : v = 62.5$, therefore $w \propto v$; *i.e.*, the weight of water varies as its volume.

One of two numbers is said to **vary inversely** as the other if the ratio of the first to the reciprocal of the second is constant. If a varies inversely as b, then $a \cdot b = k$: let pupils fully explain why.

Again, if x, y, and z are variables such that $x = kyz$, where k is a constant, then x is said to **vary jointly as** y **and** z; and if $x = \frac{ky}{z}$, then x is said to **vary directly as** y **and inversely as** z.

E.g., the time required for a railway train to travel a given distance varies inversely as the speed; for if t, r, and d represent, respectively, the time, rate, and distance, then $t \cdot r = d$, *i.e.*, $t : \frac{1}{r} = d$, where d is constant.

Again, the cost of a railway journey varies jointly as its length and its cost per mile; while the number of posts required to build a certain fence varies directly as the length of the fence, and inversely as the distance between the posts.

NOTE. It should be remarked in passing that such an expression as $w \propto v$ above (*i.e.*, the weight of water varies as its volume) is merely an abbreviated form of the proportion

$$w : w' = v : v',$$

wherein w and w' stand for the respective weights, and v and v' for the volumes, of any two quantities of water.

The theory of variation is, therefore, substantially included in that of ratio and proportion, and the only reason for even defining the expressions "varies as," "varies inversely as," etc., here, is that this convenient phraseology is so well established in physics, chemistry, etc.

EXERCISE CXXX

1. Explain and illustrate the following statements:

(1) The interest earned by a certain principal varies as the time.
(2) The circumference of a circle varies as its radius.

2. State (1) and (2) of Ex. 1 as equations (cf. § 195), also as proportions (cf. § 195, Note).

3. If $x \propto y$ and if $x = 12$ when $y = 3$, find the equation connecting x and y; also find x when $y = 7$.

Solution. Since $x \propto y$, therefore $x = ky$ where k is constant (why?); moreover, when $x = 12$ and $y = 3$, the equation $x = ky$ gives $k = 4$. Therefore, *under the given conditions*, $x = 4y$; hence, when $y = 7$, $x = 28$.

4. If $a \propto b$ and if $a = 39$ when $b = -3$, find a when $b = 2$; also when $b = \frac{1}{5}$; also find b when $a = -65$.

5. If $A \propto B$ and $B \propto C$, show that $A \propto C$.

Hint. Show that $A = kC$, where k is some constant.

6. If $m \propto n$ and $p \propto n$, prove that $m \pm p \propto n$.

7. If $3m^2 - 18 \propto 2n + 1$, and if $m = 4$ when $n = 2$, find m when $n = 23.5$.

8. The area of a circle varies as the square of its radius. If a circle whose radius is 10 ft. contains 314.16 sq. ft., find the area of a circle whose radius is 5 ft.; of one whose radius is 12 ft.

9. Find the radius of a circle whose area is twice that of a circle 10 ft. in radius (cf. Ex. 8).

10. If x varies inversely as y, how is the value of x affected if y is doubled? if y is multiplied by 10? if y is divided by -6? Explain.

11. Give three numerical examples of inverse variation.

12. If x varies inversely as y, show that:

(1) $xy = k$ (where k is constant).

(2) $x' : x'' = y'' : y'$ (where x' and y', x'' and y'' are corresponding values of the variables).

13. If x varies inversely as y, and if $x = 4$ when $y = 2$, find y when $x = -8$; when $x = 1\frac{1}{3}$; when $x = 2.5$.

14. If x varies directly as y and inversely as z, and if $x = -12$ when $y = 2$ and $z = 7$, find y when $x = 2$ and $z = 3$.

15. Solve Ex. 13 by drawing the graph of the equation connecting x and y (cf. § 141), and then measuring the y-coördinates of the points whose respective x-coördinates are -8, $1\frac{1}{3}$, and 2.5. Also show, *from the graph*, that any change in x makes an opposite change in y.

16. If the volume of a pyramid varies jointly as its base and altitude, and if the volume is 20 cu. in. when the base is 12 sq. in. and the altitude is 5 in., what is the altitude of the pyramid whose base is 48 sq. in. and whose volume is 76 cu. in.?

17. The distance (in feet) through which a body falls from a position of rest, varies as the square of the time (in seconds) during which it falls. If a body falls $257\frac{1}{3}$ ft. in 4 sec., how far will it fall in 5 sec.? how far during the 5th second? how far during the 7th second?

18. If the intensity of light varies inversely as the square of the distance from the source of light, how much farther from a lamp must a book, which is now 2 ft. away, be removed so as to receive just one third as much light?

19. The weight of a body comparatively near the earth's surface varies inversely as the square of its distance from the earth's center. Assuming that the radius of the earth is 4000 mi., find the weight of a 4-lb. brick 2000 mi. from the earth's surface. (Two solutions.)

20. The number of oscillations made by a pendulum in a given time varies inversely as the square root of its length. If a pendulum 39.1 inches long oscillates once a second, what is the length of a pendulum that oscillates twice a second?

CHAPTER XX

SERIES — THE PROGRESSIONS

196. Definitions. A **series** is a succession of numbers which proceed according to some definite law. The numbers which constitute the series are called its **terms**.

E.g., in the series 1, 2, 3, 5, 8, 13, each term after the second is the sum of the two preceding terms; in the series 2, 6, 18, 54, 162, each term after the first is 3 times the preceding term; and in the series 1, 4, 9, 16, ⋯, 81, each term is the square of the number of its place in the series.

A series which consists of an unlimited number of terms is called an **infinite series**; otherwise it is a **finite series**.

The present chapter considers only the simplest kinds of series — the so-called "progressions."

I. ARITHMETICAL PROGRESSION

197. Definitions and notation. An **arithmetical series**, or **arithmetical progression** (designated by A. P.), is a series in which the difference found by subtracting a term from the next following term is the same throughout the series. This constant difference, whether positive or negative, integral or fractional, is known as the **common difference** of the series.

E.g., the series 2, 5, 8, 11, 14, ⋯ is an A. P. whose common difference is 3. So, too, the series 18, 11, 4, -3, -10, is an A. P. whose common difference is -7.

The **elements** of any given A. P. are the first term (designated by a), the last term (l), the common difference (d), the number of terms (n), and the sum of all the terms (s).

Thus, in the series 2, 5, 8, ⋯, 32, the elements are $a = 2$, $l = 32$, $d = 3$, $n = 11$, $s = 187$.

EXERCISE CXXXI

1. Does a row of numbers written down at random constitute a series? Explain.

2. Show that 1, 7, 13, 19, 25 is an A. P. What are its elements?

3. What is d in the A. P. 7, 11, 15, 19? Extend this series four terms to the right; also three terms to the left.

4. If the 1st, 3d, and 5th terms of an A. P. are 18, 24, and 30, respectively, find d and write 8 consecutive terms of the series.

5. Write 10 consecutive terms of the series in which 19, 9, and 4 are the 1st, 5th, and 7th terms, respectively.

6. What are the elements of the A. P. 5, $5+3$, $5+6$, $5+9$, $5+12$? How is any term of this series formed from the preceding term?

7. Show that $x, x+y, x+2y, x+3y, \cdots$ is an A. P. What is d in this series? How many times must d be added to the first term to make the 2d term? to make the 3d term? the 7th term? the 10th term? the nth term?

8. Show from the definition of an A. P. that such a series may be written in the form

$$a,\quad a+d,\quad a+2d,\quad a+3d, \cdots, l-2d,\quad l-d,\quad l,$$

wherein a, d, and l represent, respectively, the first term, common difference, and last term.

198. Formulas. The elements of an A. P. are connected by the two fundamental equations (formulas) numbered (1) and (2) below.

Since each term of an A. P. may be derived by adding d to the preceding term (cf. Exs. 6–8, above), therefore, if l stands for the nth term

$$l = a + (n-1)d. \tag{1}$$

Again, since the sum of the terms of an A. P. may be written in each of the two following forms,

$$s = a + (a + d) + (a + 2d) + \cdots + (l - 2d) + (l - d) + l,$$

and $s = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a,$

therefore, by adding these equations, term by term,

$$2s = (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) + (a+l);$$

i.e., $\qquad 2s = n(a + l), \qquad$ [n terms

whence $$s = \frac{n(a+l)}{2}; \qquad (2)$$

or, substituting the value of l from (1),

$$s = \frac{n[2a + (n-1)d]}{2}.$$

NOTE. If any three of the five elements of an A. P. are given, the other two can always be found from (1) and (2) above, because, in that case, the two unknown elements will be connected by two independent equations.

Ex. 1. Find the sum of 8 terms of the A. P. $-3, -1, 1, 3 \cdots$.

SOLUTION. Here $\qquad a = -3, d = 2, n = 8;$

whence, from (1), $\qquad l = -3 + (8 - 1)2 = 11,$

and from (2), $\qquad s = \dfrac{8(-3 + 11)}{2} = 32;$

i.e., the sum of 8 terms of the A. P. is 32.

EXERCISE CXXXII

Verify formulas (1) and (2) above for the following series:

2. 10, 13, 16, 19, 22, 25, 28.

3. 26, 19, 12, 5, -2, -9, -16, -23, -30.

4. $-8, -5\frac{2}{3}, -3\frac{1}{3}, -1, 1\frac{1}{3}, 3\frac{2}{3}, 6$.

5. By means of § 198 (1), find the 17th term of 7, 11, 15, $\cdots$; then by § 198 (2), and without writing all the terms, find the sum of the first 17 terms of this series.

6. As in Ex. 5, find the 12th term of 1, 3.5, 6, 8.5, $\cdots$; also the sum of the first 8 terms.

Find the sum of:

7. Ten terms of 4, 11, 18, $\cdots$.

8. Thirty terms of -2, -0.5, 1, 2.5, $\cdots$.

9. Nineteen terms of 2, 5, 8, $\cdots$.

10. k terms of 2, 5, 8, $\cdots$.

11. n terms of $5, 5+k, 5+2k, 5+3k, \cdots$.

12. t terms of $h, 2h, 3h, \cdots$.

13. Find the sum of the even numbers from 2 to 100 inclusive. Compare your result with that in Ex. 12 when $h=2$ and $t=50$.

14. How many strokes are made in a day (24 hours) by a clock which strikes the hours only?

15. Suppose that 50 eggs are placed in a row, each 2 yd. from the next, and a basket 2 yd. beyond the last egg; how far would a boy, starting at the basket, walk in picking up these eggs and carrying them, one at a time, to the basket?

16. If a body falls 16.1 feet during the first second, 3 times as far during the next second, 5 times as far during the third second, etc., how far will it fall during the 8th second? how far during the first 8 seconds?

17. By means of § 198 (1) find n when $a=2$, $d=4$, $l=66$; also find s for this series.

18. If $a=-10$, $d=3$, $s=35$, find l and n.

HINT. Substitute in § 198 (1) and (2) and solve the resulting equations for l and n (cf. § 198, Note).

19. If $a=1$, $d=-\frac{2}{3}$, $n=9$, find l and s.

20. If $l=-\frac{19}{2}$, $n=13$, $s=-45\frac{1}{2}$, find a and d.

21. How many consecutive odd integers (beginning with 1) must be added to give the sum 225? 441? (Cf. Ex. 18.)

22. If $s=112$ and $n=7$, determine the unknown elements in the series $\cdots$, 10, 13, 16, $\cdots$, and write the series.

23. If s, n, and d are given, find a and l; *i.e.*, find a and l in terms of s, n, and d (cf. Ex. 22).

24. Find a and n in terms of d, l, and s. Make up and solve eight other examples of this kind.

25. Show that an A. P. is fully determined when any three of its elements are given.

26. If the 6th and 11th terms of an A. P. are 17 and 32, respectively, find the common difference, and also the sum of the first 11 terms.

HINT. Since the 6th term is 17, therefore $17 = a + 5d$. Similarly, $32 = a + 10d$.

27. Show that if each term of an A. P. is multiplied (or divided) by any given number, the resulting products (or quotients) are themselves in arithmetical progression.

28. If each term of an A. P. is increased or diminished by any given number, will the results be in arithmetical progression? Explain.

199. Arithmetical means. The two end terms of an A. P. are called its **extremes**, while all the other terms are called **arithmetical means** between these two.

E.g., in the A. P. 5, 9, 13, 17, 21, between the extremes 5 and 21 there are 3 arithmetical means (viz., 9, 13, and 17).

In an A. P. of 3 terms the (one) arithmetical mean between the extremes equals half their sum; for if A is the arithmetical mean between a and b, then

$$A - a = b - A, \qquad \text{[definition of an A. P.}$$

whence

$$A = \frac{a+b}{2}.$$

Ex. 1. Find the arithmetical mean between 3 and 27.

SOLUTION. The arithmetical mean between 3 and 27

$$= \frac{3+27}{2} = 15.$$

Ex. 2. Insert 5 arithmetical means between 3 and 27.

SOLUTION. In this series $a = 3$, $l = 27$, and (since there are to be 5 means) $n = 5 + 2 = 7$; whence, from § 198 (1), $d = 4$, and the required series is 3, 7, 11, 15, 19, 23, 27.

EXERCISE CXXXIII

3. Find the arithmetical mean between 14 and 9; between -5 and 17; -3 and -4; $\frac{4}{5}$ and $\frac{3}{8}$; -2.75 and 11.4.

4. Insert 4 arithmetical means between 12 and 27.

5. Insert 15 arithmetical means between 19 and 131.

6. Insert 20 arithmetical means between 16 and -40.

7. Insert 12 arithmetical means between $-\frac{3}{5}$ and $\frac{3}{5}$.

8. Insert 7 arithmetical means between $-.08$ and $-.0032$.

9. If m arithmetical means are inserted between two given numbers, such as a and b, show that the common difference of the series thus formed is $(b-a)\div(m+1)$.

10. What does the formula of Ex. 9 become when $m=1$? Is this consistent with the formula for A obtained in § 199?

11. Without actually finding the means asked for in Ex. 2, find the sum of the series formed by inserting them.

12. Find three numbers in A. P. whose sum is 15, and the sum of whose squares is 107.

HINT. Let $x-y$, x, and $x+y$ represent the required numbers.

13. Find three numbers in A. P. whose sum is 18 and whose product is $202\frac{1}{2}$.

14. The sum of the first seven terms of an A. P. is 105, and the sum of the third and fifth terms is 10 times the first term. Find the series.

15. The product of the extremes of an A. P. of three terms is 4 less than the square of the mean, and the sum of the series is 24. Find the series.

16. The sum of four numbers in A. P. is 14, and the product of the means is 12. What are the numbers?

HINT. Let $x-3y$, $x-y$, $x+y$, $x+3y$ represent the series.

17. The sum of an A. P. of five terms is 15, and the product of the extremes is 3 less than the product of the second and fourth terms. Find the series.

18. How many arithmetical means must be inserted between 4 and 25 so that the sum of the series may be 116?

19. A number, expressed by three digits in A. P., equals 30.4 times the sum of its digits; but if 9 is added to the number, the units' and tens' digits will be interchanged. Find the number.

20. In the series 1, 3, 5, $\cdots$, what is the nth term? Prove that the sum of the first n odd numbers, beginning with 1, is n^2.

II. GEOMETRIC PROGRESSION

200. Definitions and notation. A **geometric series,** or **geometrical progression** (designated by G. P.), is a series in which the quotient of each term (after the first) divided by the next preceding term is the same throughout the series. This constant quotient is called the **common ratio,** or simply the **ratio,** of the series.

E.g., the numbers 2, 6, 18, 54, $\cdots$ form a geometric series whose ratio is 3; while $\frac{2}{3}, -1, \frac{3}{2}, -\frac{9}{4}, \frac{27}{8}, \cdots$ is a G. P. whose ratio is $-\frac{3}{2}$.

The **elements** of any given G. P. are the first term (a), the last term (l), the number of terms (n), the ratio (r), and the sum of all terms (s).

E.g., in the G. P. 2, -6, 18, -54, 162, -486, 1458, $a = 2$, $l = 1458$, $n = 7$, $r = -3$, and $s = 1094$.

EXERCISE CXXXIV

Which of the following are geometric series? Explain.

1. 7, 21, 63, 189, 567.

2. 1, 4, 16, 64, 192, 576.

3. -6, 12, -24, 48, -96, 192, -384.

4. What are the *elements* of the G. P. in Ex. 3? Extend this series two terms to the right, also five terms to the left.

5. If a is the first term of a G. P., and r the ratio, what is the second term? the third? the fourth? the fifth? the fourteenth? the twenty-third? the nth? Explain.

6. Are $x, xy, xy^2, xy^3, \cdots$, and $p^4q^{-2}, p^3, p^2q^2, pq^4, q^6, p^{-1}q^8$ geometric series? If so, name a and r in each.

7. What is r in the G. P. 2, $\frac{2}{3}$, $\frac{2}{9}$, $\cdots$? in the G. P. 21, 7, $\frac{7}{3}$, $\cdots$? Are these two series merely parts of the same series? Explain.

8. If the 1st, 3d, and 6th terms of a G. P. are 12, 3, and $\frac{3}{8}$, respectively, find r and then write down the first 8 terms of the series (cf. Ex. 26, p. 311).

201. Formulas. The elements of a G. P. are connected by the two fundamental equations shown below (cf. § 198).

Since each term of a G. P. may be obtained by multiplying the preceding term by r (cf. Exs. 5 and 6, p. 313), therefore, if l represents the nth term of such a series, then

$$\boldsymbol{l = ar^{n-1}}. \qquad (1)$$

Again, if s represents the sum of a G. P. of n terms, then

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1};$$

whence, multiplying by r, we obtain

$$sr = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n,$$

and, therefore, by subtracting the second of these equations from the first, member from member,

$$s - sr = a - ar^n;$$

hence

$$\boldsymbol{s = \frac{a - ar^n}{1 - r}}. \qquad (2)$$

EXERCISE CXXXV

1. By means of § 201 (1) write down the 6th term of the geometric series 4, 12, 36, $\cdots$.

2. As in Ex. 1, write the 7th term of the G. P. 3, 6, 12, $\cdots$; then by § 201 (2) find the sum of the first 7 terms of the series. Check results by writing down and adding these first 7 terms.

3. Find the 8th term of 24, 12, 6, $\cdots$; also the 11th term of $\frac{1}{1000}$, $-\frac{1}{100}$, $\frac{1}{10}$, $\cdots$.

Find the sum of the following series:

4. 1, 2, 4, $\cdots$, to 10 terms.

5. 1, 1.5, 2.25, $\cdots$, to 6 terms.

6. 2, $-\frac{2}{3}$, $\frac{2}{9}$, $\cdots$, to 7 terms.

7. 1, $-2x$, $4x^2$, $\cdots$, to 7 terms.

8. -5, -2, $-.8$, $\cdots$, to k terms.

9. x, x^{-2}, x^{-5}, $\cdots$, to n terms.

10. Find the G. P. whose 3d term is 18, and whose 8th term is 4374.

HINT. By § 201 (1), $18 = ar^2$, and $4374 = ar^7$; hence, dividing the second of these equations by the first, $243 = r^5$, *i.e.*, $r = 3$.

11. Find the G. P. whose 5th term is $\frac{8}{3}$ and whose 9th term is $\frac{128}{243}$. Also find the sum of the first 9 terms of this series.

12. Show that § 201 (2) may be written in each of the following forms:

$$\frac{a(1-r^n)}{1-r},\quad \frac{a-rl}{1-r},\quad \frac{rl-a}{r-1},\quad \frac{ar^n-a}{r-1},\text{ and }\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

13. By actually dividing $a(1-r^n)$ by $1-r$, verify the correctness of § 201 (2).

14. If $a = 4$, $l = 972$, and $n = 6$, find r, and write the series.

15. If $n = 12$, $r = -2$, and $s = -1365$, find a and l.

16. Find the sum of a G. P. of 6 terms whose ratio is $\frac{2}{3}$ and whose last term is 32.

17. If r, n, and l are given, find a and s; that is, find a and s in terms of r, n, and l (cf. Ex. 16).

18. Find a and l in terms of n, r, and s; also r and s in terms of a, n, and l.

19. Is a G. P. fully determined when *any* three of its elements are given (cf. § 198, Note)?

20. Three numbers whose product is 216 form a G. P., and the sum of their squares is 189. What are the numbers?

HINT. Let $\frac{a}{r}$, a, and ar represent the required numbers.

21. Divide 38 into three parts which are in G. P., and such that when 1, 2, and 1 are added to these parts, respectively, the results shall be in A. P.

22. Find an A. P whose first term is 3, and such that its 2d, 4th, and 8th terms shall be in G. P.

23. If the population of the United States was 76,000,000 in 1900, and if it doubles itself every 25 years, what will it be in the year 2000?

24. Thinking $1 per bushel too high a price to pay for wheat, a man bought 10 bu., paying 3 cents for the first bushel, 6 cents for the second, 12 cents for the third, and so on. What did the tenth bushel cost him, and what was the average price per bushel?

25. Show that the amount of $\$A$ for n years at a given rate (R) of compound interest is the nth term of a G. P. whose first term is A and whose ratio is $(1 + R)$.

26. A gentleman loaned a friend $250 at the beginning of each year for 4 years. If money is worth 5 % compound interest, how much should be paid back to him at the end of the fourth year to discharge the obligation?

27. The president of a charity organization starts a "letter chain" by writing three letters, each numbered 1, requesting each recipient to remit 10 cents to the society, and also to send out 3 other letters, each numbered 2, with a similar request, the chain to close with the letters numbered 20. Should every recipient comply, how much money would be realized?

202. Infinite decreasing geometric series. A decreasing G. P. is one in which $r < 1$, numerically, and in which, therefore, the terms grow smaller and smaller as we pass from left to right in the series. Thus, 6, 3, $\frac{3}{2}$, $\frac{3}{4}$, $\frac{3}{8}$, $\cdots$ is a decreasing G. P.

If we let s_n represent the sum of the first n terms of this series, then

$$s_n = \frac{6}{1-\frac{1}{2}} - \frac{6(\frac{1}{2})^n}{1-\frac{1}{2}} \qquad \text{[Ex. 12, p. 315}$$

$$= 12 - 6(\tfrac{1}{2})^{n-1};$$

and, since $6(\frac{1}{2})^{n-1}$ approaches zero as a limit when n increases without limit [§ 194 (i)], therefore s_n approaches 12 as a limit when n becomes infinite; this is often expressed thus:

$$s_\infty = 12.$$

Similarly, whenever $r<1$, numerically, and $n=\infty$, the formula

$$s_n = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

becomes

$$s_\infty = \frac{a}{1-r};$$

since r^n approaches the limit zero as n becomes infinite.

EXERCISE CXXXVI

1. If in Ex. 2, p. 303, $AB=2$ ft., show that the successive distances traversed, when expressed in inches, form the G. P. $12, 6, 3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \cdots$.

2. By § 201 (2), find s_5 for the series in Ex. 1; also find s_8, s_9, s_{10}, and s_n.

3. From Ex. 2, p. 303, show that in the series of Ex. 1 above $s_n < 24$, no matter how large n may be. How near to 24 will s_n approach as n is made larger and larger? Explain. Also find s_∞ by § 202.

4. Find s_∞ for the series 0.6, 0.06, 0.006, $\cdots$, and thus show that $0.\dot{6}$ (*i.e.*, $0.666\cdots$) equals $\frac{2}{3}$; similarly, show that $0.\dot{1}\dot{5}$ (*i.e.*, $0.151515\cdots$) equals $\frac{5}{33}$.

Find s_∞ for each of the following series:

5. $1, -\frac{1}{2}, \frac{1}{4}, \cdots$.

6. $1, \frac{1}{3}, \frac{1}{9}, \cdots$.

7. $\frac{3}{2}, -\frac{2}{3}, \frac{8}{27}, \cdots$.

8. $\sqrt{2}, 1, \sqrt{0.5}, \cdots$.

9. $0.\dot{3}$.

10. $0.\dot{1}\dot{2}$.

11. $1.36\dot{2}$.

12. $4.7\dot{5}2\dot{3}$.

13. $1, k, k^2, \cdots$ (wherein $k<1$).

14. $x, \frac{1}{x}, \frac{1}{x^3}, \cdots$ (wherein $x>1$).

15. If, in a G. P., r is positive and less than 0.5, show that any given term of the series is greater than the sum of all the terms that follow it.

16. A point traversing a straight line moves in any given second 75 % as far as in the preceding second; if it moves 24 ft. in the first second, how far will it move before coming to rest?

17. If a sled runs 80 ft. during the first second after reaching the bottom of a hill, and if its distance decreases 20 % during each second thereafter, how far will it run on the level before coming to rest?

18. If a ball, on being dropped from a tower window 100 ft. above the pavement, rebounds 40 ft., then falls and rebounds 16 ft., and so on, how far will it move before coming to rest?

19. Although s_∞ for the series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ is 1, show that for the series $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots, s_n$ grows larger beyond all bounds, by sufficiently increasing n.

SUGGESTION. Write the series thus: $s_n = \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \cdots$, putting 8 terms in the next group, 16 in the next, and so on, and show that each group is greater than $\frac{1}{2}$.

203. Geometric means. The two end terms of a finite G. P. are called its **extremes**, while all the other terms are called **geometric means** between these two.

E.g., in the series $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \frac{2}{9}$, and $\frac{4}{27}$, the extremes are $\frac{3}{4}$ and $\frac{4}{27}$, and $\frac{1}{2}, \frac{1}{3}$, and $\frac{2}{9}$ are geometric means between them.

In a G. P. of three terms, the (one) geometric mean between the extremes equals the square root of their product; for if G is the geometric mean between a and b, then

$$\frac{G}{a} = \frac{b}{G}, \qquad \text{[definition of a G. P.}$$

whence $$G = \pm\sqrt{ab}.$$

Ex. 1. Find the geometric mean between 6 and 24; also between 10 and 8.

SOLUTION. The geometric mean between 6 and 24 is $\pm\sqrt{6 \cdot 24}$, *i.e.* ± 12; and the geometric mean between 10 and 8 is $\pm\sqrt{10 \cdot 8}$, *i.e.* $\pm 4\sqrt{5}$.

Ex. 2. Insert four geometric means between $\frac{4}{9}$ and $-\frac{27}{8}$.

SOLUTION. In this series, $a = \frac{4}{9}$, $l = -\frac{27}{8}$, and (since four means are to be inserted) $n = 4 + 2 = 6$; hence by § 201 (1), $-\frac{27}{8} = \frac{4}{9} \cdot r^5$, whence $r^5 = -\frac{243}{32}$ and $r = -\frac{3}{2}$. Therefore the required series is $\frac{4}{9}, -\frac{2}{3}, 1, -\frac{3}{2}, \frac{9}{4}, -\frac{27}{8}$.

EXERCISE CXXXVII

Find the geometric mean between the following number-pairs:

3. 18, 8. **5.** $\frac{3}{4}$, $-\frac{27}{16}$. **7.** $(a+b)$, $(a-b)^2$.

4. 5, 20. **6.** 0.5, 3.5. **8.** $2x-3$, $(x+1)^2$.

9. Insert 4 geometric means between 3 and 96.

10. Insert 3 geometric means between 2 and $\frac{1}{8}$ (two answers).

11. Insert 6 geometric means between $-3125\,x^{10}$ and $\frac{y^{-10}}{25}$.

12. If m geometric means are inserted between a and b, show that r for the series thus formed is $\sqrt[m+1]{b \div a}$.

13. What does the formula of Ex. 12 become when $m=1$? Is this consistent with the formula for G obtained in § 203?

14. Two numbers differ by 24, and their arithmetical mean exceeds their geometric mean by 6. Find the numbers.

204. Harmonic series. An **harmonic series,** or **harmonic progression** (H. P.), is a series of numbers whose reciprocals form an A. P. A supposed H. P. may therefore be tested, and problems in H. P. be solved, by an appeal to our knowledge of A. P.

E.g., the numbers $\frac{3}{7}$, $\frac{1}{4}$, $\frac{3}{17}$, $\frac{3}{22}$, $\cdots$ form an H. P. because their reciprocals, viz., $\frac{7}{3}$, 4, $\frac{17}{3}$, $\frac{22}{3}$, $\cdots$, form an A. P.

Again, if we were asked to extend the H. P. $\frac{3}{7}$, $\frac{1}{4}$, $\frac{3}{17}$, $\frac{3}{22}$, $\cdots$ one or more terms toward the right, we should need merely to form the corresponding A. P., viz., $\frac{7}{3}$, 4, $\frac{17}{3}$, $\frac{22}{3}$, $\cdots$, extend it as required (cf. Ex. 3, p. 308), and then write the reciprocals of its terms.

EXERCISE CXXXVIII

1. If the 6th term of an H. P. is $\frac{1}{3}$, and the 17th term is $\frac{2}{17}$, find the 37th term.

HINT. First find the 37th term in an A. P. whose 6th and 17th terms are 3 and $\frac{17}{2}$, respectively.

2. Insert 5 harmonic means between 2 and -3.

3. Assuming x to be the harmonic mean between a and b, show that $\frac{1}{x}-\frac{1}{a}=\frac{1}{b}-\frac{1}{x}$, and hence that $x=\frac{2ab}{a+b}$.

4. The arithmetical mean between two numbers is 5, and their harmonic mean is 3.2. What are the numbers?

5. The difference between two numbers is 2, and their arithmetical mean exceeds their harmonic mean by $\frac{1}{3}$. Find the numbers.

6. Given $(b-a):(c-b)=a:x$, prove that x equals a, b, or c, according as a, b, and c form an A. P., a G. P., or an H. P.

7. If a and b are two unequal positive numbers, and A is their arithmetical mean, G their geometric mean, and H their harmonic mean, show that: (1) $A>G>H$; and (2) $A:G=G:H$.

CHAPTER XXI

MATHEMATICAL INDUCTION — BINOMIAL THEOREM

205. Proof by induction. An elegant and powerful form of proof, and one that is very useful in many branches of mathematics, is what is known as "proof by induction."

To illustrate: suppose it to have been found by trial that $x-y$ is a factor of x^2-y^2, x^3-y^3, and x^4-y^4, and that we wish to know whether it is a factor of x^5-y^5, x^6-y^6, $\cdots$ also. Actual trial with any one of these, say $x^{25}-y^{25}$, would show that it is exactly divisible by $x-y$, but besides being somewhat tedious, this division gives no information as to whether $x-y$ is or is not a factor of $x^{26}-y^{26}$, $\cdots$ also; each successful trial increases the *probability* of the success of the next, but it *proves* nothing beyond the single case tried.

That $x-y$ is a factor of x^n-y^n, for every positive integral value of n, may be shown as follows:

Since $\qquad x^n-y^n=x(x^{n-1}-y^{n-1})+y^{n-1}(x-y),$

therefore, if $x-y$ is a factor of $x^{n-1}-y^{n-1}$, then it is a factor of the *second member* of this equation, and therefore of x^n-y^n also (why?); *i.e.*, *if* $x-y$ is a factor of the difference of any two like integral powers of x and y, then it is a factor of the difference of the *next higher* powers also.

But since, by actual trial, $x-y$ is already *known* to be a factor of x^4-y^4, therefore, by what has just been proved, it is a factor of x^5-y^5 also; again, since it is *now* known to be a factor of x^5-y^5, therefore it is a factor of x^6-y^6; and so on without end: *i.e.*, $x-y$ is a factor of x^n-y^n for every positive integral value of n.

The proof just given is an example of what is known as a proof by **mathematical induction**; such a proof consists essentially of two steps, viz.:

(1) *Showing by trial or otherwise the correctness of a given law when applied to one or more particular cases*, and

(2) *Proving that if this law is true for any given case, then it is true for the next higher case also.*

From (1) and (2) it then follows that the proposition under consideration is true for all like cases.*

EXERCISE CXXXIX

1. Prove that the sum of the first n odd integers is n^2.

SOLUTION. (1) By trial it is found that $1+3=2^2$ and $1+3+5=3^2$.

(2) Moreover, *if* $1+3+5+\cdots+(2k-1)=k^2$,

then, by adding the next odd integer to each member, we obtain

$$1+3+5\cdots+(2k-1)+(2k+1)=k^2+(2k+1)=(k+1)^2;$$

i.e., *if* the law in question is true for the first k odd integers, then it is true for the first $k+1$ odd integers also.

But, by actual trial, this law is known to be true for the first 3 odd integers, hence it is true for the first 4; and, since it is *now* known to be true for the first 4, therefore it is true for the first 5; and so on without end: hence the sum of any number of consecutive odd integers, beginning with 1, equals the square of their number.

By mathematical induction prove that:

2. $1+2+3+\cdots+n=\frac{1}{2}n(n+1)$.

3. $2+4+6+\cdots+2n=n(n+1)$.

4. $1^2+2^2+3^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$.

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* The student should carefully distinguish between mathematical induction, as here defined, and what is known as inductive reasoning in the natural sciences. A proof by mathematical induction is, from its very nature, absolutely conclusive. On the other hand, the inductive method in physics, chemistry, etc., consists in formulating a statement of a law which will fit the particular cases that are known, and regarding it as a *law* only so long as it is not contradicted by other facts not previously taken into account. From the nature of the case step (2) above cannot be applied in physics, etc.

5. $1^3+2^3+3^3+\cdots+n^3=\frac{1}{4}n^2(n+1)^2=(1+2+3+\cdots+n)^2.$

6. $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}.$

7. $1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1)=\frac{1}{3}n(n+1)(n+2).$

8. $a+ar+ar^2+\cdots+ar^{n-1}=\dfrac{a(1-r^n)}{1-r}.$

9. x^n-y^n is divisible by $x+y$ when n is even.

10. Having established (1) and (2) in the inductive proof of any law, show the generality of the law by showing that there can be no *first* exception, and therefore no exception whatever.

206. The binomial theorem. The method of induction furnishes a convenient proof of what is known as the **binomial theorem**; this theorem, which was presented without formal proof in § 112, may be symbolically stated thus:

$$(x+y)^n=x^n+\frac{n}{1}x^{n-1}y+\frac{n(n-1)}{1\cdot 2}x^{n-2}y^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3+\cdots, \tag{1}$$

wherein $x+y$ represents any binomial whatever, and n is any positive integer.

To prove this theorem by mathematical induction, observe first that it is correct when $n=2$, for it then becomes

$$(x+y)^2=x^2+\frac{2}{1}xy+\frac{2\cdot 1}{1\cdot 2}x^0y^2;\ i.e., (x+y)^2=x^2+2xy+y^2,$$

which agrees with the result of actual multiplication.

Again, if (1) is true for any particular value of n, say for $n=k$, *i.e.*, if

$$(x+y)^k=x^k+\frac{k}{1}x^{k-1}y+\frac{k(k-1)}{1\cdot 2}x^{k-2}y^2+\frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}x^{k-3}y^3+\cdots, \tag{2}$$

then, on multiplying each member of (2) by $x+y$, it becomes

$$(x+y)^{k+1}=x^{k+1}+\frac{k}{1}x^ky+\frac{k(k-1)}{1\cdot 2}x^{k-1}y^2$$
$$+\frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}x^{k-2}y^3+\cdots$$
$$+x^ky+\frac{k}{1}x^{k-1}y^2+\frac{k(k-1)}{1\cdot 2}x^{k-2}y^3+\cdots$$
$$=x^{k+1}+\frac{k+1}{1}x^ky+\left\{\frac{k(k-1)}{1\cdot 2}+k\right\}x^{k-1}y^2$$
$$+\left\{\frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}+\frac{k(k-1)}{1\cdot 2}\right\}x^{k-2}y^3+\cdots;$$

i.e., $$(x+y)^{k+1}=x^{k+1}+\frac{k+1}{1}x^ky+\frac{(k+1)k}{1\cdot 2}x^{k-1}y^2$$
$$+\frac{(k+1)k(k-1)}{1\cdot 2\cdot 3}x^{k-2}y^3+\cdots, \tag{3}$$

which is of precisely the same form as (2), merely having $k+1$ wherever (2) has k. Moreover, (3) is obtained from (2) by actual multiplication, and is therefore true if (2) is true; hence, *if the theorem is true when the exponent has any particular value* (say k), *then it is also true when the exponent has the next higher value.*

But, by actual multiplication, the theorem is *known* to be true when $n=2$, hence, by what has just been proved, it is true when $n=3$; again, since it is *now* known to be true when $n=3$, therefore it is true when $n=4$; and so on without end: hence the theorem is true for every positive integral exponent, which was to be proved.

EXERCISE CXL

1. In the expansion of $(x+y)^n$, what is the exponent of y in the 2d term? in the 3d term? in the 4th term? in the 12th term? in the rth term? What is the sum of the exponents of x and y in each term?

2. In the expansion of $(x+y)^n$ what is the largest factor in the denominator of the 3d term? of the 4th term? of the 10th term? of the rth term? In any given term, how does this factor compare with the exponent of y?

3. In the expansion of $(x+y)^n$, what is subtracted from n in the last factor of the numerator in the 3d term? in the 4th term? in the 5th term? in the 9th term? in the rth term?

4. Based upon your answers to Exs. 1–3, write down the 6th term of $(x+y)^n$. Also write the 10th term; the 17th term; and the rth term.

207. Binomial theorem continued. Strictly speaking, all that was really proved in § 206 is that, for every positive integral value of the exponent, the first *four terms* of the expansion follow the law expressed by (1); that *all* the terms follow this law will now be shown.

In multiplying (2) of § 206 by $x+y$, the 2d term of the product (3) is x times the 2d term plus y times the 1st term of (2); so, too, the 10th term of (3) would be found by adding x times the 10th term to y times the 9th term of (2), and the rth term of (3) by adding x times the rth term to y times the $(r-1)$th term of (2).

But the $(r-1)$th and the rth terms of (2) are, respectively,

$$\frac{k(k-1)(k-2)\cdots(k-r+3)}{1\cdot 2\cdot 3\cdot\cdots(r-2)}x^{k-r+2}y^{r-2}$$

and
$$\frac{k(k-1)(k-2)\cdots(k-r+3)(k-r+2)}{1\cdot 2\cdot 3\cdot\cdots(r-2)(r-1)}x^{k-r+1}y^{r-1};$$

therefore the rth term of (3) is

$$\left\{\frac{k(k-1)(k-2)\cdots(k-r+3)}{1\cdot 2\cdot 3\cdot\cdots(r-2)} + \frac{k(k-1)(k-2)\cdots(k-r+3)(k-r+2)}{1\cdot 2\cdot 3\cdot\cdots(r-2)(r-1)}\right\}x^{k-r+2}y^{r-1},$$

i.e.,
$$\frac{(k+1)k(k-1)\cdots(k-r+3)}{1\cdot 2\cdot 3\cdot\cdots(r-1)}x^{k-r+2}y^{r-1},$$

which conforms to the law for the rth term expressed by (1) of § 206. Hence the rth term, *i.e.*, every term, in (3) conforms to the law expressed by (1), which was to be proved.

EXERCISE CXLI

1. Write down the expansion of $(a+b)^5$; also of $(p-q)^8$. Explain why the alternate terms in the expansion of $(p-q)^8$ are negative.

2. Write down the 1st, 2d, 3d, and 8th terms of $(x+y)^{11}$.

3. Write down the 4th and 7th terms of $(a-x)^{13}$.

4. How many terms are there in the expansion of $(x+y)^{16}$? Write down the first three, and also the last three terms of this expansion, and compare their coefficients.

5. Write down the coefficient of the term containing a^4y^9, in the expansion of $(a-y)^{13}$.

6. Expand $(3a^2-2xy^3)^5$; compare Ex. 2, § 57.

7. Write down the 4th and 9th terms of $(\frac{3}{2}x-\frac{2}{3}y)^{11}$.

8. How many terms are there in $\left(x-\dfrac{1}{x}\right)^{18}$? Write down the 10th term. Also write down the 5th term of $\left(\sqrt{\dfrac{a}{x}}+\sqrt{\dfrac{x}{a}}\right)^8$.

9. Write down the term of $(3x^4-2x^2)^7$, *i.e.*, of $(x^2)^7(3x^2-2)^7$, which contains x^{20}.

10. Write down the term of $\left(a^3-\dfrac{2}{3a}\right)^9$ which contains a^{11}.

11. Expand $(a^{\frac{1}{2}}+3a^2x^{-1})^6$, and write the result with positive exponents.

12. Expand $(1-x+x^2)^4$ by means of the binomial theorem (cf. Exs. 40–41, p. 176).

13. By applying the law expressed in (1) of § 206 show that the coefficient of the $(n+1)$th term of $(x+y)^n$ is 1; also show that the coefficient of every term thereafter contains a zero factor, and hence that $(x+y)^n$ contains only $n+1$ terms.

14. Show that the sum of the binomial coefficients, *i.e.*, of 1, $\frac{n}{1}, \frac{n(n-1)}{2}, \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}, \cdots$, is 2^n.

HINT. After expanding $(x+y)^n$, let $x=y=1$.

15. Show that the sum of the even coefficients (*i.e.*, the 2d, 4th, ⋯) in Ex. 14 equals the sum of the odd coefficients, and that each sum is 2^{n-1}.

HINT. In $(x+y)^n$ let $x=1$ and $y=-1$.

16. Show that the coefficient of the rth term in $(x+y)^n$ may be obtained by multiplying that of the $(r-1)$th term by $\frac{n-r+2}{r-1}$, and thus show that the binomial coefficients increase numerically in going from term to term toward the center.

17. Show that the coefficient of the rth term is numerically greater than that of the $(r-1)$th term so long as $r < \frac{1}{2}(n+3)$; and thus write down the term whose coefficient is greatest in the expansion of $(x+y)^{11}$; and also in $(x+y)^{10}$.

208. Binomial theorem extended. It may be remarked in passing that the binomial theorem (§ 206), which has thus far been restricted to the case where the exponent is a positive integer, is greatly extended in Higher Algebra, where it is shown that under certain restrictions it admits negative and fractional exponents also. Although the *proof* of this fact is beyond the limits of this book, its correctness may be assumed in the following exercises.

EXERCISE CXLII

Using the binomial theorem, write the first 5 terms of:

1. $(x+y)^{\frac{1}{2}}$. **3.** $(a-c)^{\frac{2}{3}}$. **5.** $(1-as^2)^{-5}$.

2. $(1+m)^{\frac{1}{2}}$. **4.** $(a+b)^{-2}$. **6.** $(2m-k)^{-\frac{3}{4}}$.

7. Write the 6th term of $(3r-s)^{-4}$; also the 5th term of $(1-3x)^{\frac{2}{3}}$.

8. Show that in such cases as the above the binomial theorem leads to infinite series (cf. Ex. 13, p. 326).

9. Expand $(1-x)^{-1}$ to 8 terms by the binomial theorem and compare the result with the first 8 terms of the quotient $1 \div (1-x)$.

10. Show that, when expanded by the binomial theorem and simplified, $(25+1)^{\frac{1}{2}} = 5 + \frac{1}{10} - \frac{1}{1000} + \frac{1}{50000} - \cdots$; compare this result with $\sqrt{26}$ as found by the usual method.

11. By expanding $(9-2)^{\frac{1}{2}}$, find an approximate value of $\sqrt{7}$; similarly, find an approximate value of $\sqrt[3]{31}$ (*i.e.*, $\sqrt[3]{27+4}$), and of $\sqrt[5]{40}$ (*i.e.*, $\sqrt[5]{32+8}$).

209. The square of a polynomial. In § 56 it was pointed out that, by actual multiplication, the square of a polynomial consisting of 3, 4, or 5 terms equals the sum of the squares of all the terms of the polynomial, plus twice the product of each term by all those that follow it. It will now be shown that if this theorem is true for polynomials of n terms, then it is also true for those of $n+1$ terms; and from this it will follow, as in § 205, that it is true for polynomials of any finite number of terms whatever, since it is already known to be true for polynomials of five terms.

Let $a+b+c+\cdots+p+q$ be a polynomial of n terms, and let
$$(a+b+c+\cdots+p+q)^2 = a^2+b^2+\cdots+q^2+2ab+2ac+\cdots$$
$$+2aq+2bc+\cdots+2bq+\cdots+2pq.$$

In this identity replace a everywhere by $x+y$; then the number of terms in the polynomial in the first member will become $n+1$, and the second member will still consist of the sum of the squares of all the terms of the polynomial, plus twice the product of each term by all those that follow it (the student should work this out in detail); therefore, if the theorem is true for polynomials of n terms, then it is also true for those of $n+1$ terms, which was to be proved.

CHAPTER XXII

LOGARITHMS

210. Introduction. Early in the seventeenth century, two British mathematicians, Lord Napier and Henry Briggs, conceived the idea of expressing all real positive numbers as powers of 10,* arranging the exponents of these powers in a table for convenient reference, and then employing this table to simplify certain arithmetical computations, especially multiplication.

E.g., to find the product of 3.578, 7.986, and 48.67, we find from the table that

$$3.578 = 10^{0.5536},\ 7.986 = 10^{0.9023},\ \text{and } 48.67 = 10^{1.6873},$$

whence
$$\begin{aligned} 3.578 \times 7.986 \times 48.67 &= 10^{0.5536} \times 10^{0.9023} \times 10^{1.6873} \\ &= 10^{0.5536+0.9023+1.6873} \qquad [\S\ 30 \\ &= 10^{3.1432}; \end{aligned}$$

we now find from the table that

$$10^{3.1432} = 1390.6,$$

whence $\quad 3.578 \times 7.986 \times 48.67 = 1390.6.$

Thus, by performing an *addition* (of the exponents), we have found the *product* of the given numbers.

Other advantages of such a table of exponents (logarithms) will be shown later (§ 218); some necessary definitions and principles must now be given.

211. Definitions. The **logarithm** of a number (N) to any given base (b) is the exponent (x) of the power to which this base must be raised to equal the given number.

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* That it is possible to do this, either exactly or to any required degree of approximation, will be assumed in this chapter.

The logarithm of N to the base b is usually written $\log_b N$; and the two statements

$$N = b^x \quad \text{and} \quad \log_b N = x$$

are, therefore, only different ways of saying the same thing.

E.g., $\because 2^3 = 8$, $\therefore \log_2 8 = 3$; $\because 3^5 = 243$, $\therefore \log_3 243 = 5$; and $\because 10^{1.6873} = 48.67$, $\therefore \log_{10} 48.67 = 1.6873$.

EXERCISE CXLIII

1. From the equation $3^4 = 81$, find $\log_3 81$.

Translate into logarithmic equations (cf. Ex. 1):

2. $4^3 = 64$. **5.** $2^5 = 32$. **8.** $10^0 = 1$.

3. $9^2 = 81$. **6.** $(\frac{1}{2})^5 = \frac{1}{32}$. **9.** $10^{-3} = .001$.

4. $10^3 = 1000$. **7.** $2^{-5} = \frac{1}{32}$. **10.** $(\frac{1}{5})^{-3} = 125$.

Express the following statements in the exponent notation, and then verify the correctness of each:

11. $\log_7 49 = 2$. **14.** $\log_{10} 10 = 1$. **17.** $\log_3 \frac{1}{9} = -2$.

12. $\log_2 16 = 4$. **15.** $\log_{10} 1 = 0$. **18.** $\log_{10} .0001 = -4$.

13. $\log_{.5} .125 = 3$. **16.** $\log_{10} 10000 = 4$. **19.** $\log_{\frac{1}{2}} 256 = -8$.

20. Find the value of the following logarithms: $\log_3 27$; $\log_2 64$; $\log_{-8} 64$; $\log_{-6}(-216)$; $\log_4 1$; $\log_{10} .1$; $\log_{.1} 10$; $\log_{\frac{2}{5}} \frac{16}{625}$; $\log_{10} \frac{1}{10000}$; $\log_{b^2} b^6$.

21. Between what two consecutive integers does each of the following logarithms lie: $\log_{10} 83$; $\log_{10} 2224$; $\log_{10} 4$; $\log_{10} .007$; $\log_{10} .1256$? Explain your answers.

22. May the base of a set of logarithms be fractional? negative? May a logarithm itself be fractional? negative? May negative numbers have logarithms? Illustrate your answers.

212. Principles of logarithms. Since logarithms are exponents (§ 211), therefore the principles of logarithms are easily obtained from those governing exponents (§§ 171–175).

PRINCIPLE 1. *The logarithm of* 1 *to any base is* 0, *and the logarithm of the base itself is* 1; *i.e.*,

$$\log_b 1 = 0 \text{ and } \log_b b = 1.$$

The correctness of this principle follows at once from the definition of a logarithm (§ 211), and from the fact that

$$b^0 = 1 \text{ and } b^1 = b. \qquad [§§ 173, 9$$

PRINCIPLE 2. *The logarithm of a product equals the sum of the logarithms of the factors; i.e.*,

$$\boldsymbol{\log_b (MN) = \log_b M + \log_b N.}$$

For, if $M = b^x$ and $N = b^y$,

then $MN = b^{x+y}$, $[\because b^x \cdot b^y = b^{x+y}$

whence $\log_b (MN) = x + y = \log_b M + \log_b N.$

Similarly, $\log (MNP \cdots) = \log_b M + \log_b N + \log_b P + \cdots.$

Let the pupil translate Principles 3–5 below into verbal language, and prove each in detail (cf. Principle 2 above).

PRINCIPLE 3. $\boldsymbol{\log_b \frac{M}{N} = \log_b M - \log_b N.}$

HINT. If $M = b^x$ and $N = b^y$, then $M \div N = b^{x-y}$.

PRINCIPLE 4. $\boldsymbol{\log_b N^p = p \cdot \log_b N.}$

HINT. If $N = b^x$, then $N^p = (b^x)^p = b^{xp}$.

PRINCIPLE 5. $\boldsymbol{\log_b \sqrt[r]{N} = \frac{1}{r} \cdot \log_b N.}$

HINT. If $N = b^x$, then $\sqrt[r]{N} = (b^x)^{\frac{1}{r}}$. $[\sqrt[r]{N} = N^{\frac{1}{r}}$

EXERCISE CXLIV

Using Principles 1–5, express the following logarithms in terms of $\log a$, $\log c$, and $\log e$, the base b being understood throughout:

1. $\log (ac)$.
2. $\log (a^4)$.
3. $\log (ace^2)$.
4. $\log (c^2e^5)$.
5. $\log \frac{e}{c}$.

6. $\log \frac{ec}{a^3}$.

7. $\log \frac{1}{a}$.

8. $\log \sqrt{c}$.

9. $\log \sqrt[6]{ace}$.

10. $\log (a^{\frac{1}{5}}c^{\frac{2}{5}})$.

11. $\log (a^{\frac{1}{5}}c^{-\frac{2}{5}})$.

12. $\log \sqrt[3]{\frac{e^2}{a}}$.

13. $\log \sqrt[m]{ce^{-2}}$.

Express each of the following by means of a single logarithm, and explain [*e.g.*, $\log c + \log e = \log (ce)$]:

14. $\log c + \log e$.

15. $\log c - \log e$.

16. $2 \log a + 3 \log e$.

17. $4(\log c - \log a)$.

18. $\frac{1}{2}(\log e - \log 5a)$.

19. $\frac{2}{3} \log a + 4 \log 2c$.

If $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, and $\log_{10} 7 = 0.8451$, find the logarithms, to the base 10, of:

20. 6 (*i.e.*, $3 \cdot 2$).

21. 14.

22. 42.

23. 49.

24. 12 (*i.e.*, $2^2 \cdot 3$).

25. 63.

26. $\frac{3}{2}$.

27. $2\frac{1}{3}$.

28. 5 (*i.e.*, $\frac{10}{2}$).

29. 30 (*i.e.*, $3 \cdot 10$).

30. 400.

31. $\frac{7}{10}$.

32. $\sqrt{3}$.

33. $\sqrt[3]{28}$.

34. $5 \cdot 2^{\frac{1}{2}} \cdot (\frac{1}{3})^4$.

213. Common logarithms; characteristic and mantissa. Logarithms to the base 10 (called **common** or **Briggs** logarithms) possess many advantages over those having any other base, and are used in all practical computations. In the following pages the base 10 will be understood when no base is written; thus log 25 will mean $\log_{10} 25$.

Now since $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, etc., therefore $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$, $\log 1000 = 3$, etc.; and therefore the logarithm of any number between 1 and 10 lies between 0 and 1, *i.e.*, it is 0 plus a decimal; the logarithm of any number between 10 and 100 is 1 plus a decimal; the logarithm of any number between 100 and 1000 is 2 plus a decimal; etc.

Again, since $10^{-1} = .1$, $10^{-2} = .01$, $10^{-3} = .001$, etc., therefore $\log .1 = -1$, $\log .01 = -2$, $\log .001 = -3$, etc., and therefore the logarithm of any number between 1 and .1 is -1 *plus* a decimal; the logarithm of any number between .1 and .01 is -2 *plus* a decimal; etc.

The integral part (whether positive or negative) of the logarithm of a number is called the **characteristic** of the logarithm, and the decimal part (always positive) is called the **mantissa**.

E.g., log 685 is 2.8357; the characteristic is 2, and the mantissa is .8357.

214. Advantages of the base 10. (i) It follows from § 213 that the characteristic of the logarithm of any number between 10 and 100 is 1 (why?); between 100 and 1000, 2; between 10,000 and 100,000, 4; between .01 and .001, -3 (why?); between .001 and .0001, -4; etc.; *i.e.* (let pupil fully explain why),

(1) *The characteristic of the logarithm of any number greater than unity is less by one than the number of digits in its integral part;* and (2) *the characteristic of the logarithm of any number less than unity is negative, and (numerically) greater by one than the number of ciphers preceding the first significant figure of the given number.*

(ii) Another great advantage of logarithms to the base 10 is that moving the decimal point to the right or left in any given number makes no change in the mantissa of the logarithm of that number.

E.g., if $\log 57.32 = 1.7583$,

then $\log 573.2 = 2.7583$;

for $\log 573.2 = \log (57.32 \times 10) = \log 57.32 + \log 10$ [§ 212, Prin. 2

$= 1.7583 + 1 = 2.7583.$

EXERCISE CXLV

1. Which of the following logarithms have negative characteristics: log 79; log .315; log 5228; log 4.07; log .00098; log .0231; log 14865.01? Explain.

2. How many units in the characteristic of each of the above logarithms? Explain.

3. How many places to the left of the decimal point has a number if the characteristic of its logarithm is 2? 5? 0? 7? Explain your answers.

4. How many ciphers has a decimal to the left of its first significant figure if the characteristic of its logarithm is -5? -1? -3? Explain your answers.

5. If $\log 469 = 2.6712$, show that $\log 469000 = 5.6712$ and that $\log 4.69 = 0.6712$ [cf. § 214 (ii)]. Also point out the characteristic and the mantissa in each of these logarithms.

6. If $\log 8.93 = 0.9509$, find $\log .0893$; $\log 893000$; $\log 89.3$; and $\log .000893$.

7. Show that logarithms to the base 10 have at least two practical advantages over logarithms to other bases.

215. Table of common logarithms. The mantissas (*with decimal points omitted*) of the logarithms of all integers between 1 and 1000 are given in tabular form on pp. 341, 342.

This table omits the *characteristics* of the logarithms because these can be supplied by inspection (§ 214); but it includes the mantissas of the logarithms of decimal fractions as well as of integers—the mantissa of $\log 26.5$, or of $\log .00265$, for example, is the same as the mantissa of $\log 265$.

Note. Except where a number is an integral power of 10, the mantissa of its logarithm is an endless decimal. Hence the mantissas in a table are only *approximate* values, correct to three or more decimal places. The table on pp. 341, 342 is called a "four-place table" because the mantissas are computed to four decimal places. Such a table gives results less accurate than those obtained from a six-place table, for example; the degree of accuracy required in any given computation determines the choice of table.

216. Use of tables. Given a number, to find its logarithm. Write down the characteristic by § 214, *before consulting the table.* Then:

(*a*) *The number consisting of not more than three significant figures.* Find the first two figures of the number in the column headed N in the table; opposite these, and in the column headed by the third figure of the given number, find the required mantissa.

Ex. 1. Find log 374.

Solution. By § 214 the characteristic is 2. On p. 341 opposite 37 (in the N column), and in the column under 4, we find the mantissa .5729; hence log 374 is 2.5729.

Ex. 2. Find log .835.

Solution. The characteristic is -1 (§ 214). Now the mantissa of log .835 is the same as the mantissa of log 835 (§ 214), which, found as in Ex. 1, is .9217; hence log .835 is $-1 + .9217$.

Note. The mantissa being always positive, the sign of a negative characteristic is (to prevent confusion) written *over* the characteristic. Thus we write log .835 not as $-1 + .9217$, but as $\bar{1}.9217$.

Another common notation is to add 10 to the negative characteristic and then to indicate the subtraction of 10 from the entire logarithm. Thus log .835 may be written $9.9217 - 10$.

Ex. 3. Find log 6.

Solution. The characteristic is 0 (§ 214). The mantissa of log 6 is the same as that of log 600, which is .7782; hence log 6 is 0.7782.

(*b*) *The number consisting of more than three significant figures.* In this case we assume that the *logarithm* of a number varies directly as the *number* itself. While this assumption is not entirely correct (doubling 50, for example, multiplies its logarithm by 1.17 + instead of by 2), still for *small* changes in a number, it leads to results sufficiently accurate for many purposes.

Ex. 4. Find log 2547.

Solution. The characteristic is 3, and the mantissa is the same as that of log 254.7 (§ 214).

Now, mantissa of log 254 is .4048,

and mantissa of log 255 is .4065,

i.e., adding 1 to 254 adds .0017 to the mantissa of its logarithm, hence adding .7 to 254 should add, approximately, .7 of .0017, *i.e.*, .0012, to its logarithm, and hence

$$\log 2547 = 3.4048 + .0012 = 3.4060.$$

Ex. 5. Find log 74.326.

SOLUTION. The characteristic is 1, and the mantissa equals the mantissa of log 743.26, which equals (let pupil explain why) mantissa of log 743 + .26 × (log 744 − log 743).

$$= .8710 + .26 \times .0006,$$
$$= .8712;$$

hence $$\log 74.326 = 1.8712.$$

EXERCISE CXLVI

By reference to the table verify that:

6. $\log 416 = 2.6191.$

7. $\log 5 = 0.6990.$

8. $\log 83000 = 4.9191.$

9. $\log .00972 = \bar{3}.9877.$

10. $\log 5268 = 3.7216.$

11. $\log .7436 = \bar{1}.8714.$

Find the logarithm of:

12. 513.

13. 692.

14. 3.47.

15. .81.

16. 27.8.

17. .055.

18. 200.

19. 7.

20. .009.

21. 4000.

22. 36.02.

23. 6215.

24. .3972.

25. 851.3.

26. .1008.

27. 3.141.

28. 22220.

29. .000694.

30. .011111.

31. 437910.

32. .0018952.

33. Write the logarithms in Exs. 24, 26, 29, 30, and 32 in two different forms (cf. Ex. 2, Note).

217. Given a logarithm, to find the corresponding number. The number to which a given logarithm corresponds is called its **antilogarithm**. Thus,

$$\because \log 53 = 1.7243, \quad \therefore \text{antilog } 1.7243 = 53.$$

Antilogarithms are found by reversing the processes of § 216; a few examples will make the procedure plain.

Ex. 1. Find antilog $\bar{2}.5587$.

Solution. On consulting the table we find that .5587 is the mantissa of log 362, and the characteristic $\bar{2}$ tells us that there must be one cipher between the decimal point and the first significant figure [§ 214 (ii)]; hence

$$\text{antilog } \bar{2}.5587 = .0362.$$

Ex. 2. Find antilog 1.7493.

Solution. On consulting the table we find that

$$.7490 = \text{mantissa of log } 561,$$

and $$.7497 = \text{mantissa of log } 562,$$

these being the mantissas next smaller and next larger, respectively, than the given mantissa. Hence antilog 1.7493 lies *between* 56.1 and 56.2 (the characteristic being 1).

Again, since the given mantissa, viz., .7493, is $\frac{3}{7}$ of the way from .7490 to .7497, therefore the required antilogarithm is approximately $\frac{3}{7}$ of the way from 56.1 to 56.2,

i.e., $$\begin{aligned}\text{antilog } 1.7493 &= 56.1 + \tfrac{3}{7} \text{ of } 0.1\\ &= 56.1 + .043\\ &= 56.143.\end{aligned}$$

Ex. 3. Find antilog 3.1188.

Solution. $$\begin{aligned}\text{antilog } 3.1206 &= 1320,\\ \text{antilog } \underline{3.1173} &= \underline{1310}\end{aligned}$$

whence, subtracting, we obtain $.0033$ and 10

also $$3.1188 - 3.1173 = .0015;$$

therefore $$\begin{aligned}\text{antilog } 3.1188 &= 1310 + \tfrac{15}{33} \text{ of } 10\\ &= 1310 + 4.5 = 1314.5.\end{aligned}$$

EXERCISE CXLVII

Verify from the table that:

4. antilog $0.1875 = 1.54$.
5. antilog $\bar{1}.6021 = .4$.
6. antilog $1.8454 = 70.05$.
7. antilog $\bar{2}.5221 = .03328$.

Find the antilogarithm of:

8. 2.9605.
9. 0.5963.
10. 1.8451.
11. $\bar{1}.8401$.
12. 6.4983.
13. $8.0755 - 10$.

14. 3.3997. **18.** $\bar{3}.7361$. **22.** $\bar{1}.3019$.

15. $\bar{4}.2226$. **19.** 0.9002. **23.** 5.9754 – 10.

16. 2.6512. **20.** $\bar{2}.9068$. **24.** 9.5327 – 10.

17. 1.8846. **21.** 5.8049. **25.** 4.6831 – 10.

218. Computation by means of logarithms.

Ex. 1. Find p, if $p = 47.45 \times 3.514 \times .0064$.

SOLUTION

$$\log p = \log 47.45 + \log 3.514 + \log .0064; \quad [\S\ 212, \text{Prin. } 2$$

but $\log 47.45 = 1.6763,$ [§ 216

$\log 3.514 = 0.5458,$

and $\log .0064 = 7.8062 - 10,$* [§ 216, Note

therefore $\log p = 10.0283 - 10$

$= 0.0283;$

and therefore $p = 1.067.$ [§ 217

This product found in the ordinary way is .10671+.

Ex. 2. Find 3.041^4.

SOLUTION. $\log (3.041^4) = 4 \times \log 3.041$ [§ 212, Prin. 4

$= 4 \times 0.4830 = 1.9320;$ [§ 216

therefore $3.041^4 = \text{antilog } 1.9320 = 85.5.$ [§ 217

Obtained by ordinary multiplication $3.041^4 = 85.5196+$.

Ex. 3. Find $\sqrt[3]{.0572}$.

SOLUTION. $\log \sqrt[3]{.0572} = \frac{1}{3} \times \log .0572$ [§ 212, Prin. 5

$= \frac{1}{3} \times \bar{2}.7574$

$= \frac{1}{3} \times (1.7574 - 3)$ †

$= 0.5858 - 1 = \bar{1}.5858;$

therefore $\sqrt[3]{.0572} = \text{antilog } \bar{1}.5858 = .3853.$

Obtained by the method of § 120, $\sqrt[3]{.0572} = .38529+$.

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* The form 7.8602 – 10 (instead of $\bar{3}.8062$) is used for log .0064 because, in computation, negative characteristics increase the danger of errors.

† In order to divide $\bar{2}.7574$ by 3 without mixing positive and negative numbers it is well first to write $\bar{2}.7574$ in one of the following forms: 1.7574 – 3, 4.7574 – 6, 7.7574 – 9, etc., *i.e.*, to add (and then subtract) some multiple of 3 which will make the characteristic positive.

Ex. 4. If $x = \dfrac{37.22 \times (-19.86)}{(12.33)^2}$, find x.

Solution. In such examples we first find the *numerical* value of the result by regarding all the factors as positive, and then prefix the proper sign as determined by §§ 18 and 19. Thus, ignoring the minus sign, we have

$$\log x = \log 137.22 + \log 9.86 - 2 \times \log 12.33 \quad [\text{§ 212, Prin. 2 and 3}$$
$$= 1.5707 + 1.2980 - 2 \times 1.0910$$
$$= 0.6847;$$

therefore $$x = -\text{antilog } 0.6847 = -4.84.$$

Ex. 5. Given $47.5^x = 293.64$; find x.

Solution. On taking the logarithm of each member of this equation we obtain

$$x \cdot \log 47.5 = \log 293.64$$

whence $$x = \frac{\log 293.64}{\log 47.5};$$

i.e., $$x = \frac{2.4678}{1.6767} = 1.472.$$

Note. Equations in which the unknown number appears as an exponent are called **exponential** equations. Such equations cannot be solved by the methods given in the preceding pages, but are easily solved by the method illustrated in the above solution of Ex. 5.

EXERCISE CXLVIII

By logarithms find the value of:

6. 376×58.

7. 2.29×8.7.

8. $69.5 \times .00543$.

9. $-42.37 \times .236$.

10. $.2912 \times 3.141$.

11. $.0695 \times .002682$.

12. $380.7 \div 9.8$.

13. $10 \div 3.141$.

14. $3 \div 5.963$.

15. $30.07 \div .002121$.

16. $.005918 \div .0009293$.

17. $13 \times 753 \div .06238$.

By logarithms simplify:

18. 23^4.

19. $.08^{2.2}$.

20. $.395^{3.14}$.

21. $(-3.813)^7$.

22. $(1.228)^{10}$.

23. $(\frac{2}{7})^8$.

24. $(\frac{38}{47})^{24}$.

25. $(62)^{\frac{1}{3}}$.

26. $(991.7)^{\frac{1}{2}}$.

27. $(.1183)^{\frac{3}{4}}$.

28. $\sqrt{675}$.

29. $\sqrt[3]{.05001}$.

30. $\sqrt[5]{(.3192)^3}$.

31. $\sqrt[3]{.1277\sqrt[3]{17}}$.

32. $\sqrt[25]{18^{12}\sqrt{2574}}$.

33. $\dfrac{19 \times (-700)}{970 \times 1.4 \times .0616}$.

34. $\dfrac{3.141 \times .0711}{.8331 \times 51}$.

35. $\sqrt{\dfrac{.78 \times .0052 \times 16}{.339 \times 4.315}}$.

36. $\dfrac{4635^{40} \times 200.4^{\frac{1}{3}}}{10^{123}}$.

37. $\dfrac{13\sqrt[3]{11} \div 2\sqrt[3]{5}}{570\sqrt{.0121}}$.

38. $\dfrac{2^{\frac{1}{2}} \times (\frac{1}{2})^{\frac{2}{3}} \times \sqrt[3]{\frac{3}{2}}}{(\frac{1}{3})^{1.63}}$.

39. If a, b, and c are the sides of a triangle, and s is one half their sum, the area of the triangle is $\sqrt{s(s-a)(s-b)(s-c)}$. Find, by logarithms, the area of the triangle whose sides are 13.6 ft., 15.1 ft., and 20.1 ft.; also the area of the triangle whose sides are 260 ft., 319 ft., and 464 ft.

Solve for x (cf. Ex. 5):

40. $16^x = 354$.

41. $7^x = 9.59$.

42. $28.8^x = 12750$.

43. $6^x = 5^{x+1}$.

44. $2^{5x} = 11^{3x+1}$.

45. $152^{\log x} = 3275$.

46. From (1), § 201, show that in a G. P. $\log r = \dfrac{\log l - \log a}{n-1}$; also find r when $a = 10$, $n = 10$, and $l = 196830$.

47. If A is the amount of P dollars at r % compound interest for n years, show that $A = P(1+r)^n$; also solve this equation for each letter it contains. (Cf. Ex. 25, p. 316, also Ex. 46 above.)

48. Find the amount of \$700 for 5 years at 4 % compound interest; also the amount of \$450 for 10 years at 3 % compound interest.

49. In what time will \$800 amount to \$1834.50 if put at compound interest at 5 %?

Table of Common Logarithms

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	0	1	2	3	4	5	6	7	8	9

Table of Common Logarithms

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

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